Creativity in Mathematics through Analysis of ill-defined Problems

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Abstract

Groups of vocational students are introduced to mathematical modelling as simplifications of real world situations. Problems which have been tackled successfully include: population modelling for a nature reserve, modelling the spread of an infectious disease, traffic speed control by road humps, and modelling traffic flow through a road tunnel. This approach simulates the work of professional mathematicians, and can promote students' creativity and motivation in mathematics.

Key words

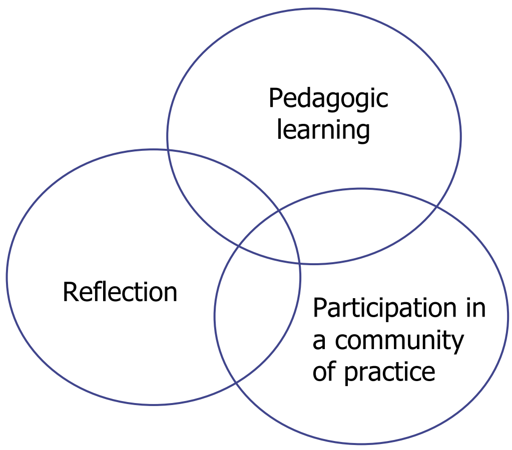
numeracy, modelling, motivation, creativity, spreadsheet, population model, traffic flow model, epidemic model

Introduction

It is unfortunate that traditional school mathematics teaching often restricts the creativity of students. Problems must be tackled using standard algorithms with little opportunity for imaginative input, and problems have clearly defined correct answers. These tasks are very different to the ill-defined mathematical problems often experienced by real mathematicians.

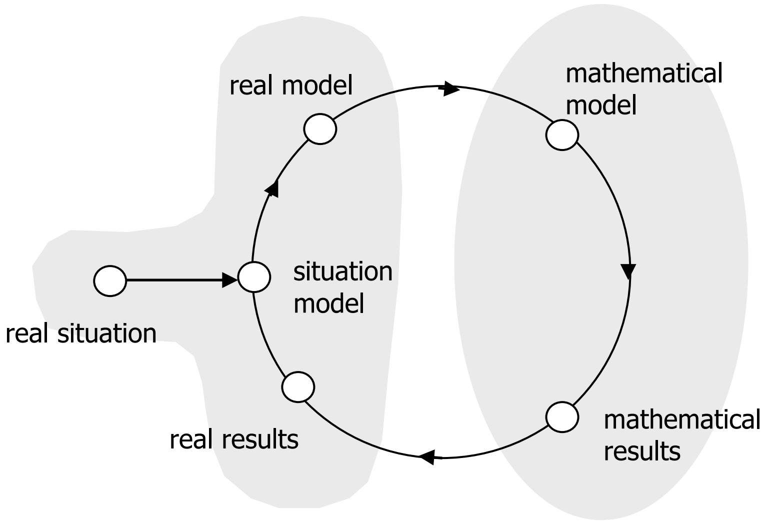
A project has been carried out with computing and engineering students at a further education college in Wales, to examine an alternative approach to learning mathematics. Students creatively devise models of real life situations from first principles using simple mathematics. This approach is consistent with the concept of practical numeracy. We might define numeracy as a combination of any appropriate mathematical techniques, plus the skills needed to apply these effectively in a real world context. Non-mathematical components of numeracy might include: background knowledge of the problem context, a common sense approach to problem solving, and the ability to communicate well.

The approach used in this project has been based on a model of adult learning consisting of three components which are often combined:



Pedagogic learning describes formal study in which a teacher provides the core knowledge required in a subject. ***Participation in a community of practice*** refers to the way in which students learn from experienced practitioners within a practical apprentice relationship (Lave and Wenger, 1991; Eraut et al, 1998). ***Reflection*** describes the way in which students make sense of events and learn through experience (Schön, 1983; Boud and Walker, 1998).

Mathematical modelling provides an opportunity to progress from pedagogic learning towards a realistic simulation of participation in the community of professional users of mathematics. Modelling, through its handling of ill-defined systems, allows opportunities for reflection, team working and problem solving. A useful theoretical framework around which to plan and conduct mathematical modelling activities has been provided by Blum and Leiß (Keune and Henning, 2003):



This schematic sequence can be illustrated by some modelling examples:

Owls and mice population model

Students were introduced to a scenario in which a nature reserve was being created to conserve a population of Tawny Owls. The task for the group was to use mathematical modelling to advise on the management of the reserve in order to ensure the long term survival of the owl population.

Students worked in small groups to discuss and reach agreement on the factors to be included or excluded from the model. Interest centred on the food supply for the owls, which is dominated by mice.

A real model was set up to show the relationships which exist between the populations of owls and mice. No algebraic symbols are introduced at this stage. It was deduced that:

mice next year	=	increased population due to breeding	-	decrease due to owl predation
owls next year	=	fall in population if food supply is short	+	increase if mice available

A mathematical model can now be formulated. The symbols **M** and **T** were introduced to represent the average numbers of mice and tawny owls per hectare. A subscript is used to represent the year, with **n** being the current year, and **n+1** representing next year. Assuming a 10% increase in mouse population if there is no owl predation, the students were able to derive an expression:

$$M_{n+1} = 1.1\ M_n - k\ M_n T_n$$

The number of mice eaten by owls was seen to depend on both the number of mice in the area, and also on the number of owls which were searching for prey. A parameter **k** was necessary to represent the chances of a mouse being caught when spotted by an owl. As a first guess, this was given a value of 0.05.

A similar relationship was derived for the number of tawny owls per hectare, assuming a 20% fall in owl numbers over the year if no mice were available as a food supply :

$$T_{n+1} = 0.8\ T_n + 0.005\ M_n T_n$$

The mathematical problem was then solved using a spreadsheet. The formulae for mice and owls are interlinked, allowing populations of each species to be calculated if the populations are known from the previous year. It is simply necessary to provide starting populations for year 0. An example section of spreadsheet showing the formulae is:

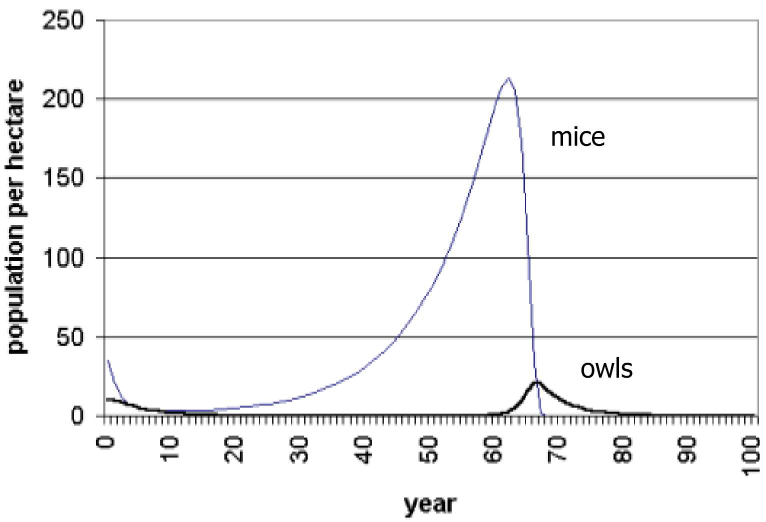
	A	B	C	D
1		year	mice	owls
2		0	35	10
3		=B2+1	=1.1*C2-0.05*C2*D2	=0.8*D2+0.005*C2*D2

$M_{n+1} = 1.1\ M_n - 0.05\ M_n T_n$ $T_{n+1} = 0.8\ T_n + 0.005\ M_n T_n$

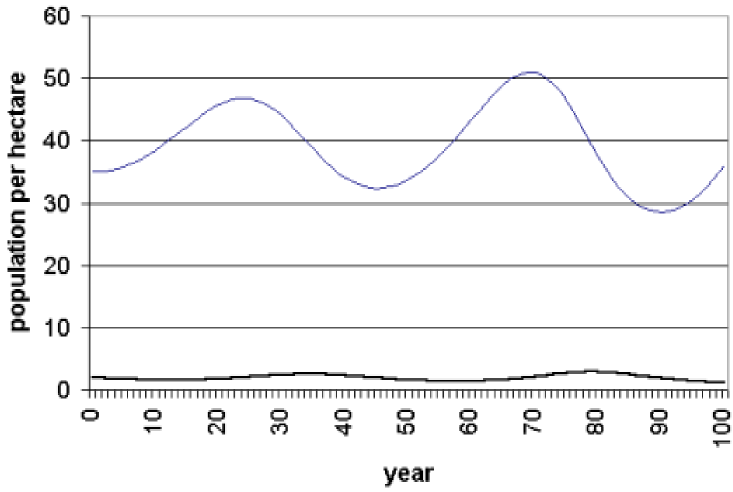
The formulae can then be copied downwards for the required number of years.

The next modelling stage is to interpret the mathematical results. For starting numbers of 35 mice and 10 tawny owls per hectare in year 0, it is discovered that the populations are unstable.

After an initial decline in both mice and owl numbers, the mouse population shows an exponential increase due to a low predation rate. Owls return to the area in increasing numbers, producing a massive increase in predation. With their food supply depleted, the owls finally leave and do not return.



By careful adjustment of the numbers of owls and mice in year 0, the students were able to find stable conditions which allowed the long term survival of the owl population. They were then able to formulate advice to the managers of the nature reserve as to the numbers of owls to introduce.



A final stage of the modelling cycle would be to monitor the performance of the model in predicting real world events, and modify the relationships and parameter values as necessary.

Modelling the spread of an infectious disease

Students were asked to use mathematical modelling to investigate the progress of a non-fatal epidemic through a human population. This type of modelling is important for planning health service resources, and to predict effects on the workforce through illness and absence.

Students were able to identify three groups within the population:

- Susceptible – those able to catch the illness
- Infected – those able to transmit the illness
- Recovered – those who cannot catch or transmit the illness again

We begin with a statement of the model in descriptive terms:

change in the number of susceptible individuals = number infected

This can be formulated as:

$$S_n = S_{n-1} - r\, S_{n-1} I_{n-1}\, \Delta t$$

where the n subscript notation is again used to represent time periods, this time of duration **Δt**. The number of persons becoming infected depends on both the number of susceptible individuals and the number of infectious persons with whom they come into contact.

Similar relations can be derived for the number infected and the number who have recovered:

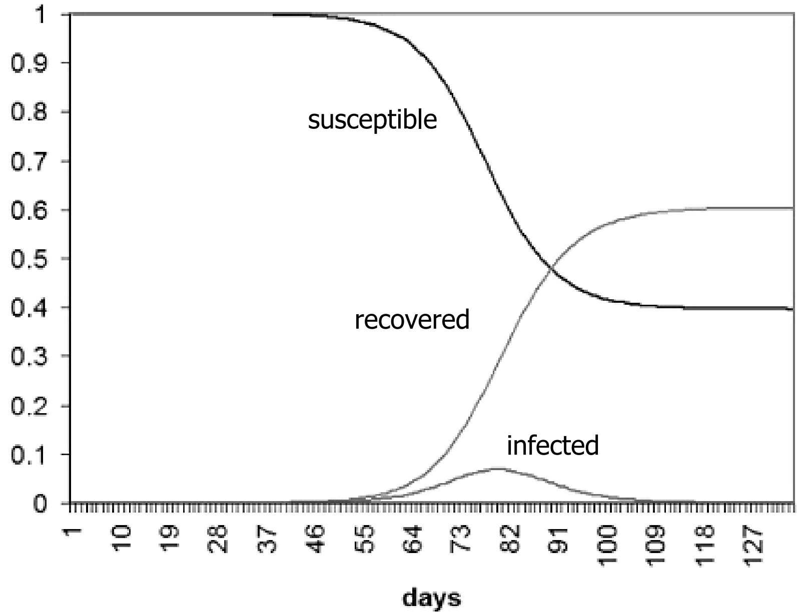
change in the number of infected individuals = number newly infected - number newly recovered

$$I_n = I_{n-1} + (\, r\, S_{n-1} I_{n-1} - k\, I_{n-1}\,)\, \Delta t$$

change in the number of recovered individuals = number newly recovered

$$R_n = R_{n-1} + k\, I_{n-1}\, \Delta t$$

Typical results for an epidemic model are shown in the graph alongside. We might note that a substantial number of susceptible people remain after the epidemic has passed. They simply do not come into contact with infected individuals, so remain healthy.



The parameter **r** used in the previous equations has a special significance. It is known as the *epidemiological parameter* (Keeling, 2001) and is a measure of the infectiousness of the illness. Typical values are:

AIDS	r = 2 to 5	Measles	r = 16 to 18
Smallpox	r = 3 to 5	Malaria	r >100

Traffic flow modelling for a road tunnel

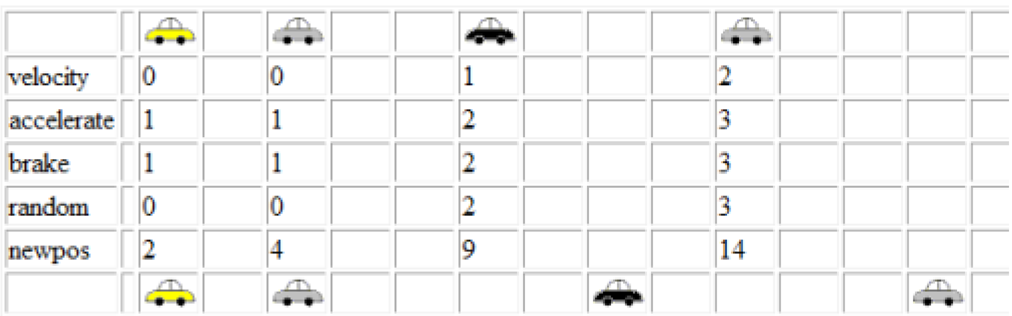
In the next model, students were asked to consider the flow of traffic through a proposed road tunnel. The objective is to advise the highway authority on how to maintain the maximum safe vehicle flow through the tunnel.

Students were able to identify a series of rules (Vandaele et al., 2000) which could be applied to each vehicle at any particular time step:

- Rule 1: acceleration
  All cars that have not already reached the maximum speed allowed in the tunnel will accelerate to this velocity v_{max}
- Step 2: safety distance
  A car will reduce speed if necessary to maintain a safe distance from the car in front.
- Step 3: randomisation
  Drivers often behave unpredictably, reducing speed more than is necessary for the traffic conditions. This excess breaking occurs randomly within the traffic flow.

After applying these rules, the new velocity $\mathbf{v_n}$ for each car is calculated, and the car is moved forward for the start of the next time step.

As an interesting way of displaying the model, students developed a simple computer animation of car movements using the programming language JAVASCRIPT:



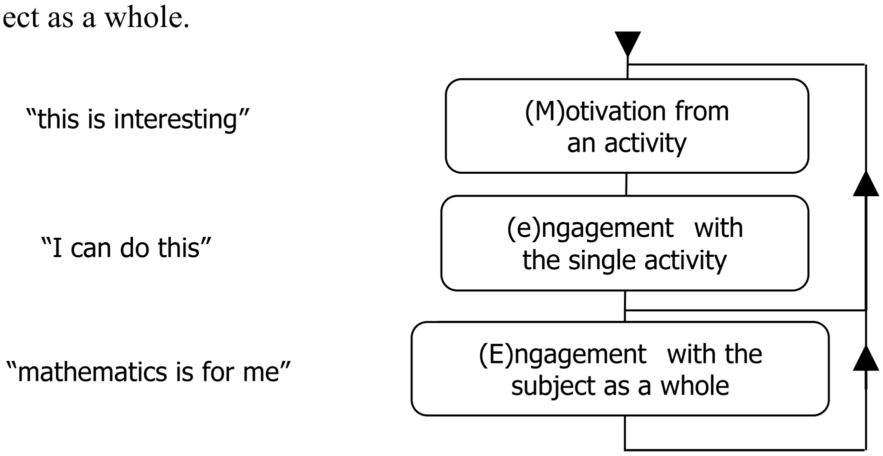
velocity	0		0			1				2				
accelerate	1		1			2				3				
brake	1		1			2				3				
random	0		0			2				3				
newpos	2		4			9				14				

Results from the model demonstrate that the total traffic flow through the tunnel is dependent on the pattern of vehicle arrivals at the tunnel entrance. Flow is maximal for a steady arrival rate. If vehicles arrive in tight bunches separated by long gaps, this can lead to unsteady flow through the tunnel and periods of stationary traffic. Advice to the highway authority would be to maintain steady flow on the approach roads to the tunnel, perhaps through the construction of roundabouts.

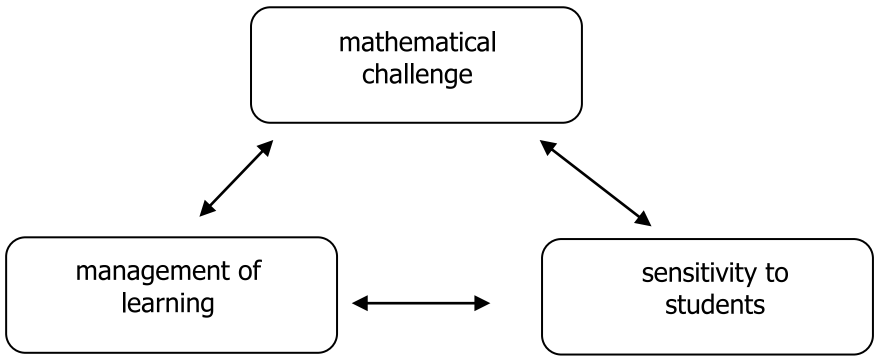
The value of mathematical modelling

The use of the types of mathematical model described above, in conjunction with new technology, has proved to be a powerful combination for motivating young people. Students have developed skills in numeracy, problem solving and working with others, in addition to improving communication skills.

The approach taken has demonstrated the value of the MeE theory of Martin (2002). Students are motivated to engage in a series of interesting challenges as a step towards engaging with the subject as a whole.



The objective of this project has been to simulate the activities of professional mathematicians, so that students experience the range of learning modes which might occur in a real work environment (Eraut et al., 1998).   Jaworski (1992) has proposed a triad of components for effective mathematics teaching:



In the context of mathematical modelling activities, ***mathematical challenge*** is achieved by providing students with a real life scenario in which they can act as a participant in a community of practice. ***Management of learning*** allows the teacher to organise whole group discussions and feedback by individual students as a means of encouraging reflection. ***Sensitivity to students*** allows the teacher to recognise opportunities for presenting new mathematical methods and skills through pedagogic learning.

References:

Boud, D and Walker, D. (1998) 'Promoting reflection in professional courses: the challenge of context', *Studies in Higher Education*, 23 (2).

Eraut, M., Alderton, J., Cole, G. and Senker, P. (1998) 'Learning from other people at work', in F. Coffield (ed.) *Learning at Work*. Bristol: The Policy Press.

Jaworski B. (2003) Teaching development through classroom research and its relation with/to 'Design Research'. *Proc.British Society for Research into Lerning Math*. 23(2), 31-36.

Keeling M. (2001). The mathematics of diseases. *Plus magazine*, issue 44.

Keune M. and Henning H. (2003). 'Modelling and Spreadsheet Calculation', in Ye Q.-X., Blum W., Houston K. and Jiang Q.-Y. (eds) *Mathematical Modelling in Education and Culture: ICTMA 10*. Horwood Publishing, Chichester, UK.

Lave, J. and Wenger, E. (1991) 'Legitimate peripheral participation in communities of practice', in J. Lave and E. Wenger, *Situated Learning*. Cambridge: Cambridge University Press.

Martin A.J. (2002) Motivation and academic resiliance: Developing a model of student enhancement. *Australian Journal of Education*, 47, 88-106 .

Vandaele N, Van Woensel T. and Verbruggen A. (2000). A queueing based traffic flow model. *Transport Research-D: Transport and environment*, v5,no.2.