Mechanics: calculus for motion in a straight line

Learning points

In previous sections, the equations of motion were introduced. These can be applied when motion takes place in a straight line and acceleration is constant:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

where v is final velocity, u is initial velocity, a is acceleration, t is time, and s is distance travelled.

The first two of these equations can readily be obtained using calculus. For constant acceleration a:

$$velocity = \int a\,.dt \quad = \quad at + c$$

The velocity calculated by the integral is the final velocity v, whilst the constant of integration is the initial velocity u.

$$v = u + at$$

The second equation is derived in a similar way:

$$distance\ travelled = \int v\,.dt$$

$$s = \int u + at\,.dt \quad = \quad ut + \frac{1}{2}at^2$$

No constant of integration is needed in this case. The distance travelled s represents the ***difference*** between the initial position and the final position of the moving object, rather than its absolute position at the end of the time interval.

The third equation of motion can be obtained by combining the first two equations using algebra:

$v = u + at,$ so $t = \frac{v-u}{a}$

Substituting for t in $s = ut + \frac{1}{2}at^2$ gives:

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

$$s = \frac{uv - u^2}{a} + \frac{1}{2}\left(\frac{v^2 - 2uv + u^2}{a}\right)$$

$$as = uv - u^2 + \frac{1}{2}v^2 - uv + \frac{1}{2}u^2$$

Cancelling opposite terms:

$$as = \frac{1}{2}v^2 - \frac{1}{2}u^2$$

Doubling, and rearranging:

$$v^2 = u^2 + 2as$$

Variable acceleration

When using calculus, we are not restricted to acceleration being constant. For example:

> A car accerates from a stationary position along a straight road. The acceleration (ms^{-2}) is given by the equation:
>
> $$a = 0.4t + 1.2$$
>
> Find an expression for the distance travelled (m) after t seconds.

As before, we can integrate the equation for acceleration to obtain an equation for velocity.

$$v = \int (0.4\,t + 1.2)\ .dt \quad = \quad 0.2\,t^2 + 1.2\,t + c$$

No constant of integration is needed, as the car began to accelerate from a stationary position.

Integrating a second time to obtain an expression for distance travelled:

$$s = \int v\,.dt \quad = \quad \int (0.2\,t^2 + 1.2\,t)\ .dt$$

$$s = \frac{2}{30}t^3 + \frac{3}{5}t^2$$

The sequence using acceleration to determine velocity and then distance travelled can be reversed by using differentiation. For example:

> A train leaves a station and the distance travelled (m) in t seconds is given by the equation:
>
> $$s = 0.1\,t^3 + 0.2\,t^2 + 0.8\,t$$
>
> Find an expression for the acceleration of the train (ms^{-2}) after t seconds.

We begin by differentiating the equation for distance travelled to obtain an equation for velocity:

$$v = \frac{ds}{dt} \quad = \quad \frac{d}{dt}\,(0.1\,t^3 + 0.2\,t^2 + 0.8\,t)$$

$$v = 0.3\,t^2 + 0.4\,t + 0.8$$

The velocity equation is then differentiated to obtain an equation for acceleration:

$$a = \frac{dv}{dt} \quad = \quad \frac{d}{dt}\,(0.3\,t^2 + 0.4\,t + 0.8)$$

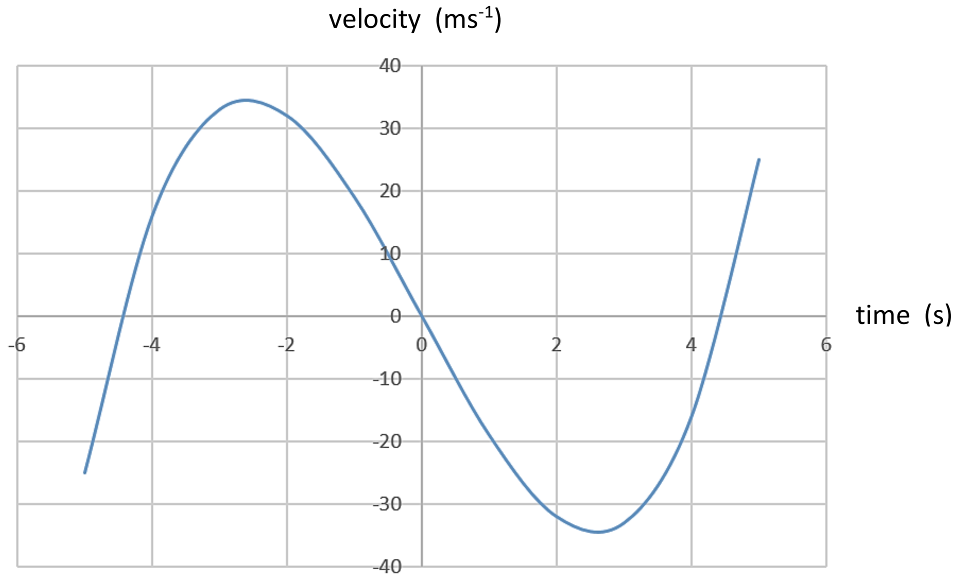
$$a = 0.6\,t + 0.4$$

Graphical representation

A better understanding of the calculus process can be gained by representing functions graphically. Consider the situation:

> A trolly moves backwards and forwards along a test track. Its velocity is given by:
>
> $$v = t^3 - 20t \qquad\qquad -5 \le t \le 5$$
>
> over the period from 5 seconds before to 5 seconds after a reference time.

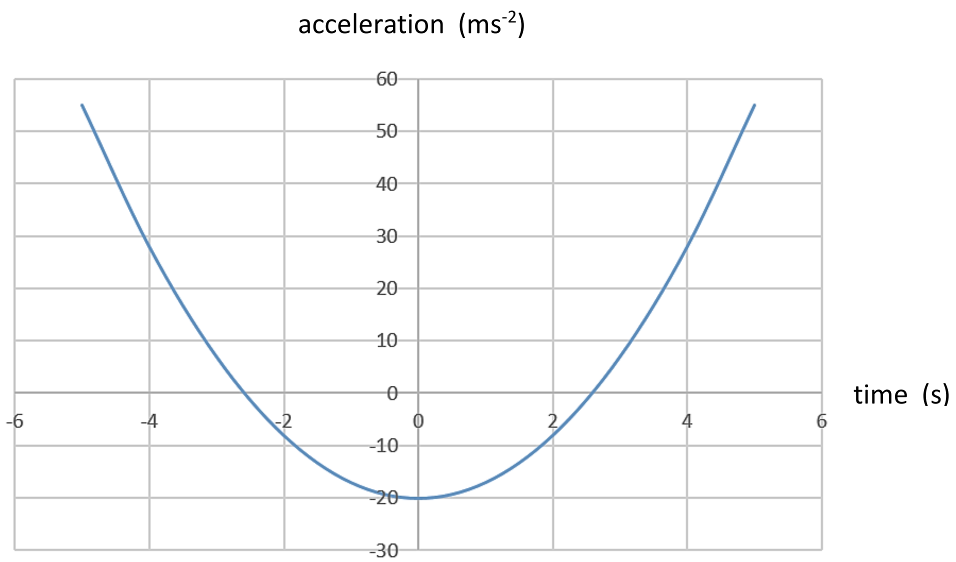
A graph of velocity against time can be plotted:



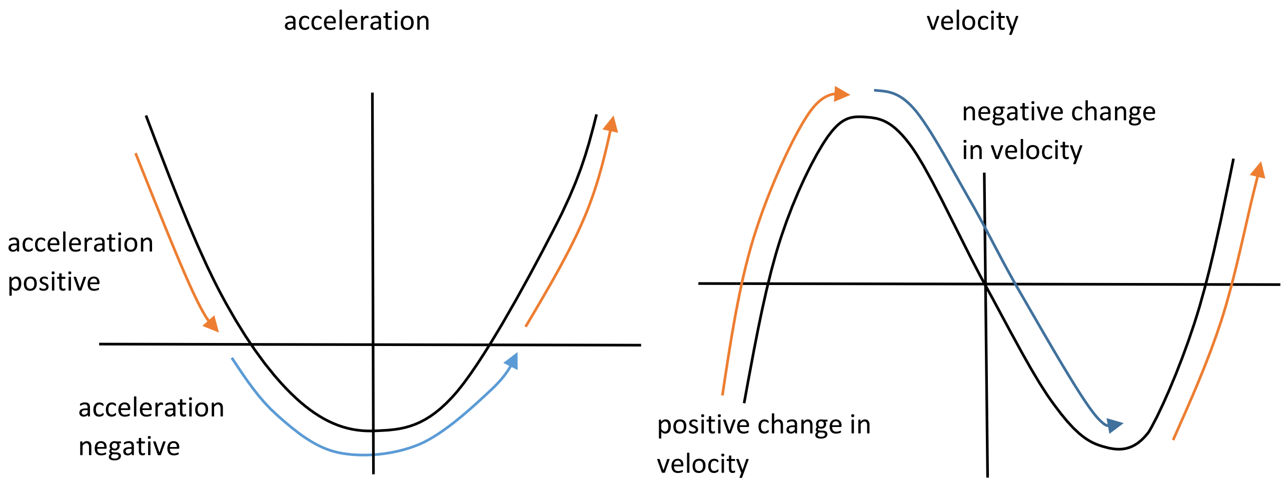
A function for the corresponding acceleration can be obtained by differentiation:

$$a = \frac{dv}{dt} = 3t^2 - 20$$

Acceleration initially has a large positive value. This falls to zero then becomes negative before returning to zero and increasing to a large positive value again.



Points on the acceleration-time curve specify the gradient $\frac{dv}{dt}$ of the corresponding points on the velocity-time curve.



- Initially the acceleration is positive. This causes velocity to increase in the positive direction.
- Acceleration decreases to zero, and the velocity momentarily becomes constant at the maximum point.
- Acceleration becomes negative, causing the velocity to fall to zero then increase in the negative direction.
- Acceleration returns to zero, and the velocity momentarily becomes constant at the minimum point.
- Acceleration again becomes positive. This causes velocity to increase again in the positive direction.

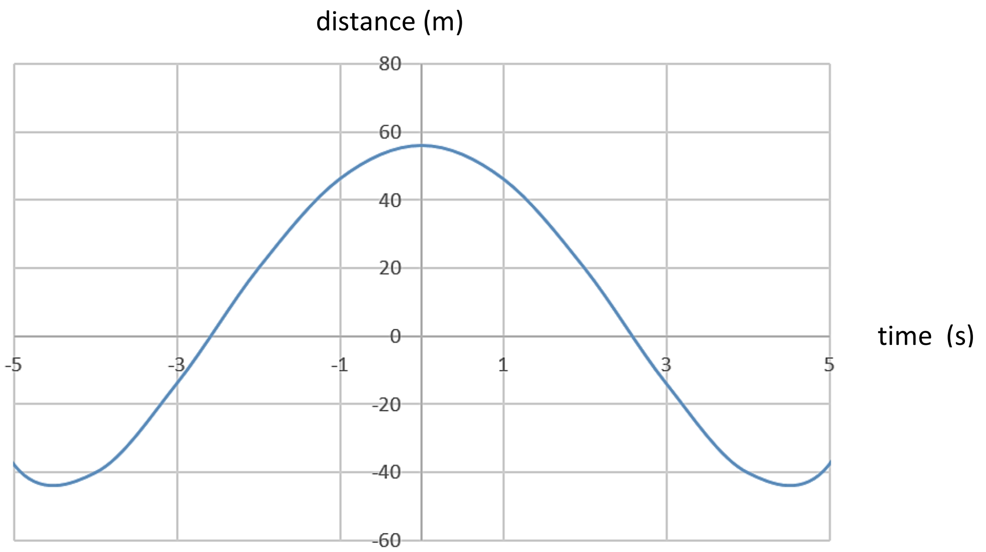
Suppose now that we use the velocity function to determine the distance travelled, s, by the trolley during the time period. This can be done by integration:

$$s = \int t^3 - 20t \,.\, dt \quad = \quad \left[\frac{1}{4}t^4 - 10t^2\right] + c$$

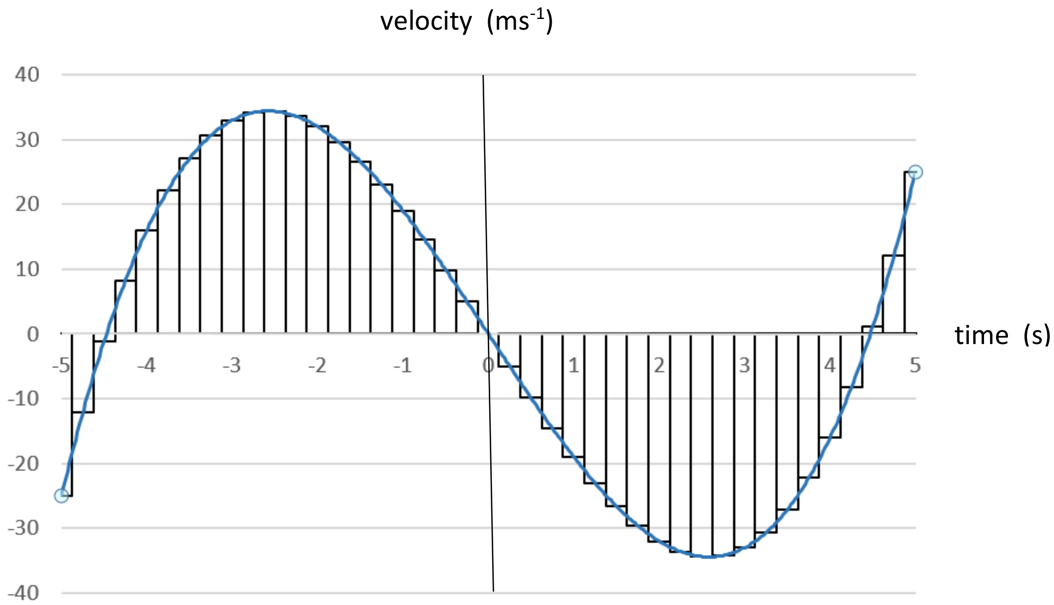
where c is a constant of integration. If it is known that at time t = - 4 seconds, the trolley was at a position 40 metres in the negative-x direction from the origin, then c can be calculated:

$$s = \frac{1}{4}t^4 - 10t^2 + 56$$

A distance-time graph can be plotted. This is again consistant with the velocity-time graph. At t = -4 the trolley is located 40 metres in the negative direction from the origin. Between t = -4 and t = 0, the velocity is positive so the trolley moves towards and past the origin. After t = 0 the velocity becomes negative. The trolley stops momentarily then begins to move back to the origin and then beyond in the negative x-direction.



The distance travelled is given by the area between the velocity-time curve and the x-axis. We can approximate this area by a series of rectangular blocks. Choosing a narrow block width of 0.01 gives a reasonably accurate approximation of 212.8 for the total area.



The objective of integration is to produce an exactly accurate result, as if the areas of an infinite number of infinitely thin columns could be added. This is done by means of a definite integral, but care is needed.

We might begin by trying to calculate the complete area between t = -5 and t = 5:

$$s = \int_{-5}^{5} t^3 - 20t\,.dt \;\; = \;\; \left[\frac{1}{4}t^4 - 10t^2\right]_{-5}^{5}$$

Evaluating this directly gives an answer of 0, which is clearly incorrect! The reason is that the areas on the negative and positive sides of the y-axis have been subtracted from each other, rather than being added.

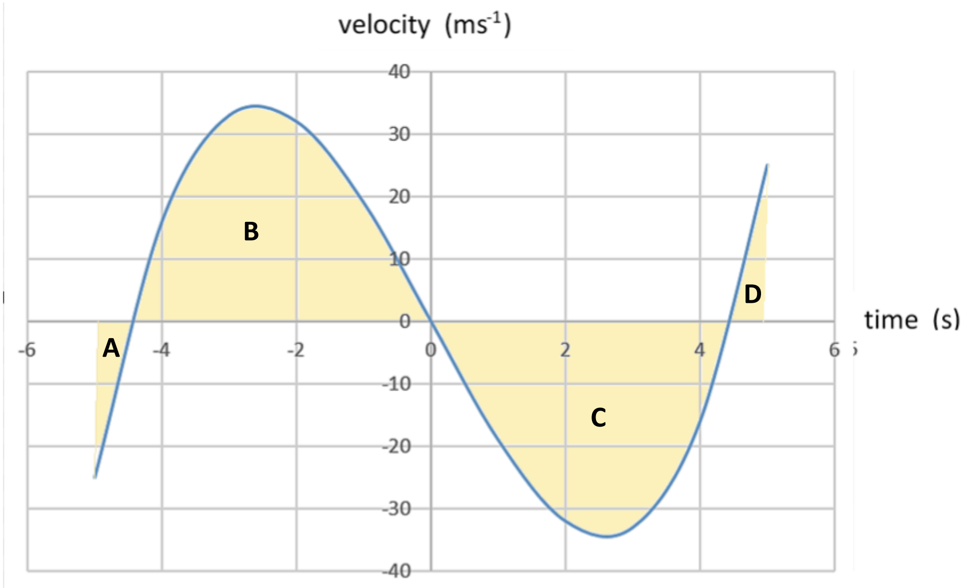
If the two areas are evaluated separately:

$$\left[\frac{1}{4}t^4 - 10t^2\right]_{-5}^{0} \qquad and \qquad \left[\frac{1}{4}t^4 - 10t^2\right]_{0}^{5}$$

we now get a total of 187.5, which is much closer to the estimate of 212.8. However, a significant difference remains. There is actually another problem present:

The curve crosses the x-axis at – 4.47 and again at 4.47. The areas above and below the x-axis are given different +/- signs, so the totals produced are added incorrectly.

It is actually necessary to find the four separate sub-areas marked A, B, C and D below, then add these to obtain the final total:



When this is done, a result of 212.5 is found which is very close to the earlier estimate of 212.8

Practical applications

In previous sections we have discussed the use of calculus to analyse the motion of cars, trains and aircraft. However, calculus is used in solving a far more extensive range of motion problems. Some examples are given below.

Motion of bodies in space

Complex calculations are needed to determine the acceleration, velocity and distance travelled by bodies in space such as space craft and meteorites. As objects move close to planets, gravitational forces can affect their motion. Calculations involving calculus are used to determine the flight paths of meteorites and to predict possible impacts with Earth.

Flowing water

Calculus is used widely in computer programs to model the flow of water in rivers, and tidal flows in estuaries.

River flow modelling is used by engineers to ensure that bridges will withstand maximum expected water flows during floods, and will not be damaged by scouring around the bridge supports.

Estuary water flow modelling is linked to the prediction of storms. This allows warnings of tidal surges to be issued, so that tidal gates such as the Thames Barrier can be closed to prevent flooding.

Meteorology

Calculus is used extensively in metorological forecasting. An important application is the calculation of future wind speeds and directions based on current patterns of atmospheric pressure. Winds will carry air from areas of high pressure to low pressure, but the paths taken will be affected by other factors such as the rotation of the Earth, and the positions of hills and mountains.

Prediction of high wind speeds is important for the safety of ships and boats at sea. On land, high wind speeds can be dangerous for high sided vehicles. Bridges across estuaries, such as the Severn Bridge, may be closed to large vehicles when high winds are expected.

Multiple choice questions

1.

A particle P moves along the horizontal x-axis.
Its velocity v ms^{-1} at time t seconds is given by

$$v = 12t - 3t^2.$$

Given that the particle is at the origin O when $t = 1$, find an expression for the displacement of the particle from O at time t s.

Select an answer

A: $3t^2 - t^3 - 5$ B: $3t^2 - 3t^3 - 8$

C: $6t^2 - t^3$ D: $6t^2 - t^3 - 5$

Correct answer: D

$$v = 12t - 3t^2$$

Intergating to find an expression for the distance travelled:

$$s = \int 12t - 3t^2 . dt \ = \ 6t^2 - t^3 + c$$

When t = 1, the distance s = 0:

$$6(1)^2 - (1)^3 + c, \quad so \quad c = -6 + 1 \ = \ -5$$

The expression for displacement of the particle is:

$$s = 6t^2 - t^3 - 5$$

2.

A particle P is moving on the x axis and its displacement from the origin, x m, t seconds after a given instant, is given by

$$x = \frac{1}{3}t\left(t^2 - 3t - 24\right),\ t \geq 0.$$

Determine the displacement of P when it is instantaneously at rest.

Select an answer

A: 16.75 B: -26.67 C: 29.33 D: -38.44

Correct answer: B

$$x = \frac{1}{3}t(t^2 - 3t - 24)$$

$$= \frac{1}{3}t^3 - t^2 - 8t$$

The particle will be momentarily at rest when its velocity $v = 0$

$$v = \frac{dx}{dt} = t^2 - 2t - 8$$

When v = 0, $t^2 - 2t - 8 = 0$

$$(t - 4)(t + 2) = 0$$

The particle motion is only defined for $t \geq 0$, so the particle is stationary at t = 4 seconds.

The displacement at t = 0 is:

$$\frac{1}{3}(4)^3 - (4)^2 - 8(4) = \frac{64}{3} - 16 - 32$$

$$= -26.67$$

3.

A particle P is moving on the x axis and its velocity v ms^{-1}, t seconds after a given instant, is given by

$$v = t^2 - 4t - 12,\ t \geq 0.$$

Find the acceleration of P when $t = 3$.

Select an answer

A: $2\ ms^{-2}$ B: $4\ ms^{-2}$ C: $-3\ ms^{-2}$ D: $-2.5\ ms^{-2}$

Correct answer: A

$$v = t^2 - 4t - 12$$

Differentiating to find an expression for acceleration:

$$a = \frac{dv}{dt} = 2t - 4$$

When t = 3:

$$a = 2(3) - 4 = 2\ ms^{-2}$$

4.

A particle P is moving on the x axis and its velocity v ms^{-1}, t s after a given instant, is given by

$$v = t^2(3-t),\ t \geq 0.$$

When $t = 2$, P is observed to be 4 m from the origin O, in the positive x direction.

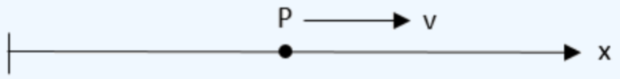
The particle is at instantaneous rest initially, and when $t = T$.

Determine the distance of P from O when $t = T$.

Select an answer

A: $2\frac{1}{2}$ m B: $3\frac{5}{8}$ m C: $6\frac{3}{4}$ m D: 6 m

Correct answer: C



$$v = t^2(3-t) \quad = \quad -t^3 + 3t^2$$

The displacement of P is given by:

$$s = \int v.dt \quad = \quad \int -t^3 + 3t^2.dt$$

$$s = -\frac{1}{4}t^4 + t^3 + c$$

When t = 2, s = 4 metres in the positive x-direction. Substituting for s and t:

$$4 = -\frac{1}{4}(2)^4 + (2)^3 + c$$

$$4 = -4 + 8 + c,$$

$$4 + 4 - 8 = c \quad so \quad c = 0$$

Thus, the expression for displacement is

$$s = -\frac{1}{4}t^4 + t^3$$

Since $v = t^2(3-t)$, the particle is stationary when $t^2 = 0$ and when $(3-t) = 0$. This occurs at t = 0 and t = 3.

When t = 3, the position of the particle is:

$$-\frac{1}{4}(3)^4 + (3)^3$$

$$= -20\frac{1}{4} + 27 = \quad 6\frac{3}{4} \text{ metres,}$$

5.

A car, starting from rest, is driven along a horizontal track.

The velocity of the car, $v\,\text{m s}^{-1}$, at time t seconds, is modelled by the equation

$$v = 0.48t^2 - 0.024t^3 \text{ for } 0 \le t \le 15$$

Find the distance the car travels during the first 10 seconds of its journey.

Select an answer

A: 150 m B: 100 m C: 160 m D: 80 m

Correct answer: B

$$v = 0.48\,t^2 - 0.024\,t^3$$

It is assumed that the car is moving in the positive x-direction during the first 10 seconds of the journey. This allows the distance to be calculated as a definite integral:

$$s = \int_0^{10} (0.48\,t^2 - 0.024\,t^3).\,dt$$

$$= [0.16\,t^3 - 0.006\,t^4]_0^{10}$$

Evaluating the integral:

$$s = [0.16\,(10)^3 - 0.006\,(10)^4] - [0]$$

$$= 100\,m$$

6.

A particle P is moving on a straight line.

At time t seconds, the distance of P from a fixed origin O is x metres and its acceleration is

$$(8-2t)\ \text{ms}^{-2}$$

in the direction of x increasing.

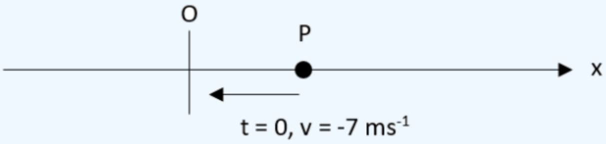
It is further given that when $t=0$, P was moving towards O with speed $7\ \text{ms}^{-1}$.

Determine the total distance covered by P in the first 7 seconds.

Select an answer

A: $69\frac{2}{3}$ m B: $44\,m$ C: $47\frac{1}{4}$ m D: $39\frac{1}{3}$ m

Correct answer: D



Velocity is found by integrating the acceleration formula:

$$v = \int (8 - 2t).dt \quad = \quad -t^2 + 8t + c$$

The velocity is -7 ms^{-1} at t = 0, so c = -7. The expression for velocity is:

$$v = -t^2 + 8t - 7$$

$$v = (t - 1)(-t + 7)$$

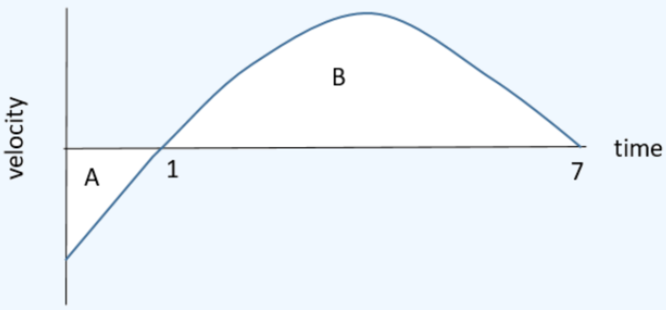
The velocity is zero and the particle is momentarity stationary at t = 1 and t = 7.

Acceleration is given by $a = 8 - 2t$.

At t = 1, $a = 8 - 2 = 6$. The acceleration is positive and the velocity is increasing in the positive x-direction.

At t = 7, $a = 8 - 14 = -6$. The acceleration is negative and the velocity is increasing in the negative x-direction.

There is therefore a velocity maximum betweeen t = 1 and t = 7



The distance travelled is given by the area between the velocity curve and the x-axis. This should be calculated as the two separate areas A and B, one below and the other above the axis, to avoid an addition error.

Total distance travelled $= \int_0^1 (-t^2 + 8t - 7)\,.\,dt \quad + \quad \int_1^7 (-t^2 + 8t - 7)\,.\,dt$

$$\left[-\frac{1}{3}t^3 + 4t^2 - 7t\right]_0^1 \quad + \quad \left[-\frac{1}{3}t^3 + 4t^2 - 7t\right]_1^7$$

$$= \left(\left[-\frac{1}{3} + 4 - 7\right] - [0]\right) \quad + \quad \left(\left[-\frac{1}{3}(7)^3 + 4(7)^2 - 7(7)\right] - \left[-\frac{1}{3} + 4 - 7\right]\right)$$

Taking both areas as positive:

$$s \;= 3\frac{1}{3} \quad + \quad \left[-\frac{343}{3} + 196 - 49\right] - \left[-3\frac{1}{3}\right]$$

$$= 3\frac{1}{3} \; + \; 36 \quad = 39\frac{1}{3}$$

7.

Peter is driving through the countryside, along a straight horizontal road at a constant speed of 22.5 ms^{-1}.

He sees a fallen tree blocking the road ahead, at a distance of 75 m ahead, so he immediately applies the brakes trying to stop his car before it hits the fallen tree.

The way he applies the brakes is such so that the **deceleration** of his car is given by $\left(3+\frac{1}{4}t\right) \text{ ms}^{-2}$, where t is measured since the instant he first applied the brakes.

Peter's car stops D m **before** he hits the tree.

Determine the value of D.

Select an answer

A: 11 m B: 3 m C: 9 m D: 2 m

Correct answer: B

The deceleration is $\left(3+\frac{1}{4}t\right) ms^{-2}$, so the velocity decreases by this amount each second.

The velocity is given by:

$$v = \int a\,.dt \quad = \quad \int -\frac{1}{4}t - 3\,.dt \quad = \quad -\frac{1}{8}t^2 - 3t + c$$

At t = 0, the veleocity of the car is 22.5 ms^{-1} :

$$22.5 = -\frac{1}{8}(0)^2 - 3(0) + c, \quad so \quad c = 22.5$$

The velocity of the car in the seconds after the brakes are applied is:

$$-\frac{1}{8}t^2 - 3t + 22.5$$

When the car stops, v = 0 so:

$$-\frac{1}{8}t^2 - 3t + 22.5 = 0$$

Solving by means of the quadratic equation $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a = -\frac{1}{8}, b = -3, c = 22.5$

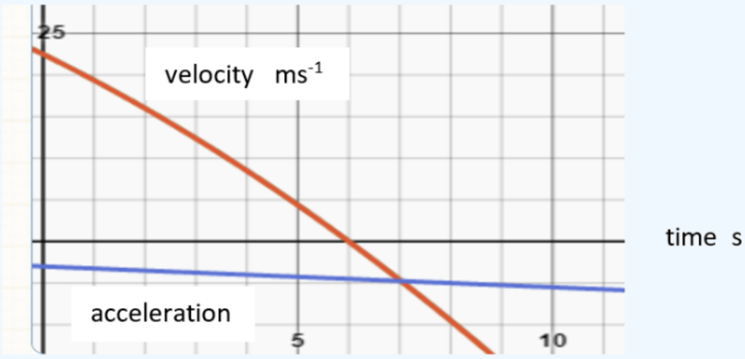
$$t = \frac{-(-3) \pm \sqrt{9 - 4\left(-\frac{1}{8}\left(22\frac{1}{2}\right)\right)}}{-\frac{1}{4}}$$

$$t = \frac{3 \pm \sqrt{9 + 11.25}}{-\frac{1}{4}}$$

$$t = \left(3 \pm \sqrt{20.25}\right) \times (-4)$$

$$t = (3 \pm 4.5) \times (-4)$$

Calculated solutions are t = 6 seconds and t = -30 seconds. The solution t = 6 is the valid solution.



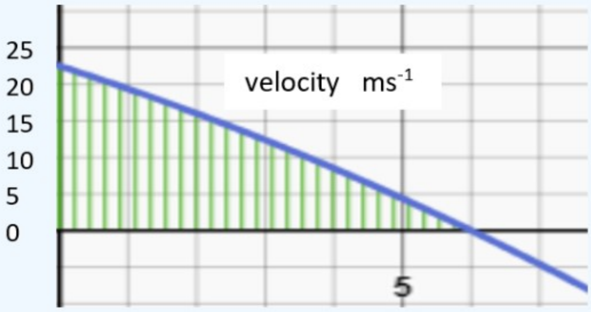
In 6 seconds, the distance travelled is given by:

$$s = \int_0^6 -\frac{1}{8}t^2 - 3t + 22.5 \ .dt$$

$$= \left[-\frac{1}{24}t^3 - \frac{3}{2}t^2 + 22\frac{1}{2}t\right]_0^6$$

$$= \left[-\frac{1}{24}(3)^3 - \frac{3}{2}(3)^2 + 22\frac{1}{2}(3)\right] - [0]$$

$$= -9 - 54 + 135 \ = \ 72\,m$$



The car stopped 3 m before the fallen tree.

8.

A particle P is moving on the x axis and its velocity v ms^{-1}, t seconds after a given instant, is given by

$$v = \begin{cases} 6t - t^2 & 0 \le t \le 5 \\ 25 - 4t & t > 5 \end{cases}$$

Find the greatest speed of P for $0 \le t \le 6$ seconds

Select an answer

A: $9\,ms^{-1}$ B: $12\,ms^{-1}$ C: $5\,ms^{-1}$ D: $15\,ms^{-1}$

Correct answer: A

The velocity is given by two functions. The first operates in the interval $0 \le t \le 5$:

$$v = \quad 6t - t^2$$

The acceleration during the period $0 \le t \le 5$ is given by:

$$a = \frac{dv}{dt} = \quad 6 - 2t$$

The acceleration is momentarily zero at t = 3.

Before t = 3, the acceleration is positve and the velocity is increasing.

After t = 3, the acceleration is negative and the velocity is decreasing.

The stationary point at t = 3 is a velocity maximum.

At t = 3, the velocity is:

$$v = 6(3) - (3)^2 \quad = \quad 18 - 9 \quad = \quad 9\,ms^{-1}$$

At t = 5. $(6t - t^2) = 5$ and $(25 - 4t) = 5$. There is a smooth transition between the functions.

The function which operates from t = 5 onwards:

$$v = 25 - 4t$$

is a negative linear function with acceleration

$$\frac{dv}{dt} = -4$$

The velocity continues to decrease over the period $5 \le t \le 6$

The greatest speed of the particle occurs at t = 3, when the velocity is 9 ms^{-1}

Longer exam questions

1.

A particle P is moving on the x axis and its acceleration a ms^{-2}, t seconds after a given instant, is given by

$$a = 8 - 2t\,,\; t \ge 0\,.$$

Initially, P is on the positive x axis 84 m away from the origin O, and is moving towards O with a speed of 7 ms^{-1}.

a) Find an expression for the velocity of P.

b) Calculate the maximum velocity of P.

c) Determine the times when P is instantaneously at rest.

d) Show that when $t = 12$, P is passing through O.

(a) The velocity of the particle is given by:

$$v = \int a\,.dt \quad = \quad \int 8 - 2t\,.dt$$

$$v = -t^2 + 8t + c$$

When t = 0, v = -7 ms^{-1}. Therefore:

$$-7 = -(0)^2 + 8(0) + c, \quad \text{so} \quad c = -7$$

The velocity of the particle is given by:

$$v = -t^2 + 8t - 7$$

(b) The acceleration is zero when $8 - 2t = 0$, so a = 0 when t = 4 seconds.

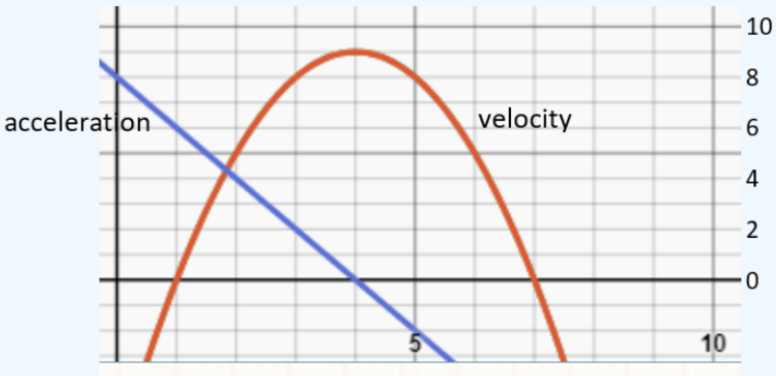
When t < 4, the acceleration is positive, so the particle velocity is increasing.

When t > 4, the acceleration is negative, so the particle velocity is reducing.

The maximum velocity is at time t = 4. The velocity at this point is:

$$v = -(4)^2 + 8(4) - 7 \quad = \quad -16 + 32 - 7 \quad = \quad 9\,ms^{-1}$$

Checking by graph plot:



(c)

$$v = -t^2 + 8t - 7$$

$$= (-t + 7)(t - 1)$$

so v = 0 when t = 1 or t = 7 seconds.

(d) The displacement of the particle is given by:

$$s = \int v\,.dt \quad = \quad \int -t^2 + 8t - 7\,.dt$$

$$s = -\frac{1}{3}t^3 + 4t^2 - 7t + c$$

When t = 0, s = 84 m. Therefore:

$$84 = -\tfrac{1}{3}(0)^3 + 4(0)^2 - 7(0) + c, \quad \text{so} \quad c = 84$$

$$s = -\frac{1}{3}t^3 + 4t^2 - 7t + 84$$

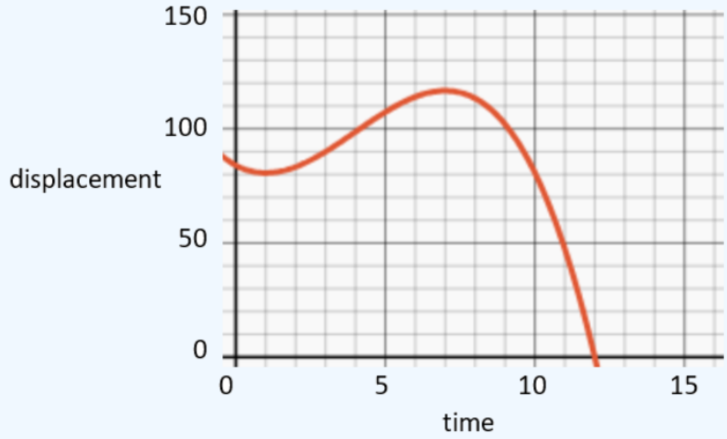
When t = 12 seconds,

$$s = -\frac{1}{3}(12)^3 + 4(12)^2 - 7(12) + 84$$

$$s = -4(144) + 4(144) - 84 + 84 \quad = \quad 0$$

At the time t = 12, the particle is passing through the origin x = 0.

Checking by graph plot:



2.

A particle, P, is moving along a straight line such that its acceleration $a\ \text{m s}^{-2}$, at any time, t seconds, may be modelled by

$$a = 3 + 0.2t$$

When $t = 2$, the velocity of P is $k\ \text{m s}^{-1}$

(a) Show that the initial velocity of P is given by the expression $(k - 6.4)\ \text{m s}^{-1}$

[4 marks]

(b) The initial velocity of P is one fifth of the velocity when $t = 2$

Find the value of k.

[2 marks]

(a) Velocity is given by:

$$v = \int a\,.dt \quad = \quad \int 3 + 0.2t\,.dt$$

$$v = 0.1\,t^2 + 3t + c$$

When t = 2, the velocity has the value k:

$$0.1\,(2)^2 + 3(2) + c = k$$

$$0.4 + 6 + c = k, \quad \text{so} \quad c = k - 6.4$$

When t = 0:

$$v = 0.1\,(0)^2 + 3(0) + (k - 6.4)$$

$$= (k - 6.4)\ ms^{-1}$$

(b) Comparing values for velocity when t = 0 and t =2:

$$k - 6.4 = \frac{1}{5}(k)$$

$$5(k - 6.4) = k$$

$$4k = 32 \quad \text{so} \quad k = 8$$

3.

A particle, P, is moving in a straight line with acceleration a m s^{-2} at time t seconds, where

$$a = 4 - 3t^2$$

(a) Initially P is stationary.

Find an expression for the velocity of P in terms of t.

[2 marks]

(b) When $t = 2$, the displacement of P from a fixed point, O, is 39 metres.

Find the time at which P passes through O, giving your answer to three significant figures.

Fully justify your answer.

[5 marks]

(a) Velocity is given by:

$$v = \int a \, .dt = \int 4 - 3t^2 \, .dt$$

$$v = -t^3 + 4t + c$$

When t = 0, v = 0, therefore c = 0. The expression for velocity is: $v = -t^3 + 4t$

(b) Displacement is given by:

$$s = \int v \, .dt = \int -t^3 + 4t \, .dt$$

$$s = -\frac{1}{4}t^4 + 2t^2 + c$$

When t = 2, s = 39, so:

$$39 = -\frac{1}{4}(2)^4 + 2(2)^2 + c$$

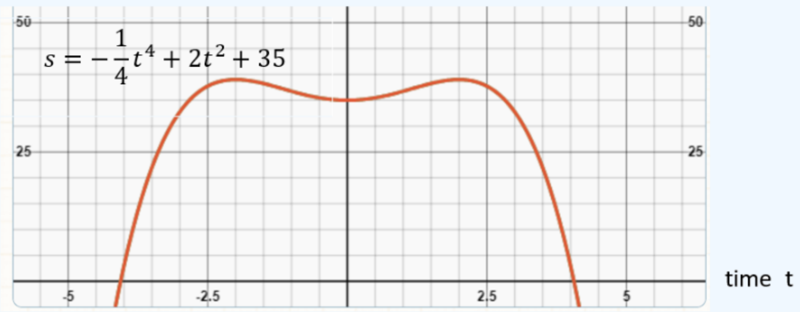
The equation for displacement is: $s = -\frac{1}{4}t^4 + 2t^2 + 35$

The equation can be solved by the method of successive approximation, locating the time at which the displacement changes from a positive to negative value:

t	s	
0	35.00	
4	3.00	low
8	-861.00	high
6	-217.00	high
5	-71.25	high
4.5	-27.02	high
4.2	-7.51	high
4.1	-2.02	high
4.05	0.54	low
4.07	-0.47	high
4.06	0.04	low
4.065	-0.21	high

The solution lies between 4.060 and 4.065. The solution to three significant figures is therefore 4.06

Checking by graph plot:



4.

A particle P moves along a straight line.

At time t seconds, the velocity $v\,\text{m s}^{-1}$ of P is modelled as

$$v = 10t - t^2 - k \qquad t \geqslant 0$$

where k is a constant.

(a) Find the acceleration of P at time t seconds.

(2)

The particle P is instantaneously at rest when $t = 6$

(b) Find the other value of t when P is instantaneously at rest.

(4)

(c) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 6$

(4)

(a) Acceleration is given by:

$$a = \frac{dv}{dt} = \frac{d}{dt}(10t - t^2 - k)$$

$$a = 10 - 2t$$

(b) When t = 6, v = 0. Substituting in the expession for velocity: $v = 10t - t^2 - k$

$$10(6) - (6)^2 - k = 0$$

$$k = 60 - 36 = 24$$

The equation for velocity is: $v = 10t - t^2 - 24$

$$v = (t - 6)(-t + 4)$$

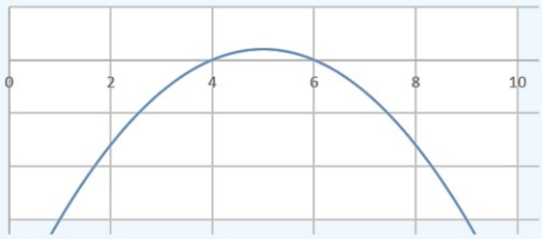
Velocity is zero when t = 4 seconds and when t = 6 seconds.

(c) The displacement of the particle is given by:

$$s = \int v\ .dt = \int -t^2 + 10t - 24\ .dt$$

$$s = -\frac{1}{3}t^3 + 5t^2 - 24t + c$$

The velocity function is a quadratic with a negative square term, so tits curve is a parabola with a maximum. The curve cuts the x-axis at t = 4 and t = 6:



The distance travelled in the interval $0 \le t \le 6$ is given by the total of the area below the x-axis from t = 0 to t = 4, and the area above the x-axis from t = 4 to t = 6. These need to be evaluated separately to avoid an incorrect subtraction.

$$\left[-\frac{1}{3}t^3 + 5t^2 - 24t\right]_0^4 = \left[-\frac{1}{3}(4)^3 + 5(4)^2 - 24(4)\right] - 0$$

$$= -37\frac{1}{3}$$

$$\left[-\frac{1}{3}t^3 + 5t^2 - 24t\right]_4^6 = \left[-\frac{1}{3}(6)^3 + 5(6)^2 - 24(6)\right] - \left(-37\frac{1}{3}\right)$$

$$= 1\frac{1}{3}$$

The total distance travelled is therefore: $37\frac{1}{3} + 1\frac{1}{3} = 38\frac{2}{3}\ metres$

5.

A particle P moves along the x-axis so that its velocity v ms^{-1} at time t seconds $(t \geqslant 0)$ is given by

$$v = 3t^2 - 24t + 36.$$

a) Find the values of t when P is instantaneously at rest. [2]

b) Calculate the total distance travelled by the particle P whilst its velocity is decreasing. [7]

(a) Velocity is given by:

$$v = 3t^2 - 24t + 36$$

$$= (3t - 18)(t - 2)$$

Velocity is zero when t = 2 seconds and t = 6 seconds.

(b) Acceleration is given by:

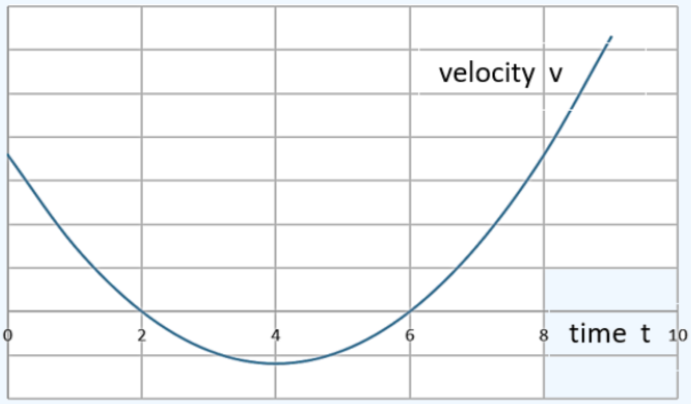
$$a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 - 24t + 36)$$

$$a = 6t - 24$$

When t < 4, acceleration is negative, so velocity is decreasing.

When t > 4, acceleration is positve, so velocity is increasing.

The velocity curve has a minimum at t = 4.



The velocity of the particle is decreasing between t = 0 and t = 4.

The displacement s of the particle is given by:

$$s = \int v\,.dt \quad = \quad \int 3t^2 - 24t + 36\,.dt$$

$$s = t^3 - 12t^2 + 36t + c$$

The distance travelled by the particle is given by the area between the velocity curve and the x-axis. This should be evaluated as two separate areas: above the axis between t = 0 and t = 2, and below the axis between t = 2 and t = 4. This avoids incorrect addition of the two areas.

$$[t^3 - 12t^2 + 36t]_0^2 = [(2)^3 - 12(2)^2 + 36(2)] - 0$$

$$= 8 - 48 + 72 \quad = \quad 32$$

$$[t^3 - 12t^2 + 36t]_2^4 = [(4)^3 - 12(4)^2 + 36(4)] - 32$$

$$= 64 - 192 + 144 - 32 = -16$$

The total distance travelled between t = 0 and t 4 is therefore: 32 + 16 = 48 metres.

6.

A car is travelling on a straight horizontal road with constant velocity of 37.5 ms^{-1}.

The driver applies the brakes and the car decelerates at $(9.25 - t)$ ms^{-2}, where t s is the time since the instant when the brakes where first applied.

a) Show that while the car is decelerating its velocity is given by

$$\tfrac{1}{4}\left(2t^2 - 37t + 150\right) \text{ ms}^{-1}.$$

b) Hence find the time taken to bring the car to rest.

c) Determine the distance covered while the car was decelerating.

(a) The velocity is given by:

$$v = \int a \ .dt = \int t - 9.25 \ .dt$$

$$v = \frac{1}{2}t^2 - 9.25\,t + c$$

When t = 0, v = 37.5 ms^{-1}, so c = 37.5

$$v = \frac{1}{2}t^2 - 9.25\,t + 37.5$$

$$v = \frac{1}{4}(2t^2 - 37t + 150)\,ms^{-1}$$

(b) Factorising the expression for v:

$$v = \frac{1}{4}(2t - 25)(t - 6)$$

Thus v = 0 when t = 6 or t = 12.5. The car first comes to a halt after 6 seconds.

(c) The distance travelled is given by:

$$s = \int_0^6 v \, .dt = \int_0^6 \left(\frac{1}{2}t^2 - 9\frac{1}{4}t + 37\frac{1}{2}\right) .dt$$

$$s = \left[\frac{1}{6}t^3 - \frac{37}{8}t^2 + \frac{75}{2}t\right]_0^6$$

$$= \left[\frac{1}{6}(6)^3 - \frac{37}{8}(6)^2 + \frac{75}{2}(6)\right] - 0$$

$$= 36 - 166\frac{1}{2} + 225 \quad = \quad 94.5\ m$$

7.

A particle moves along the horizontal x-axis so that its velocity $v\,\text{ms}^{-1}$ at time t seconds is given by

$$v = 6t^2 - 8t - 5.$$

At time t = 1, the particle's displacement from the origin is -4 m. Find an expression for the displacement of the particle at time t seconds. [3]

The displacement is given by:

$$s = \int v \, .dt \quad = \quad \int (6t^2 - 8t - 5) \, .dt$$

$$s = 2t^3 - 4t^2 - 5t + c$$

When t = 1, s = -4 metres. Substituting in the expression for displacement:

$$-4 = 2(1)^3 - 4(1)^2 - 5(1) + c$$

$$-4 - 2 + 4 + 5 = c, \quad \text{so } c = 3$$

The displacement of the particle at time t is: $2t^3 - 4t^2 - 5t + 3$

8.

A particle is moving in a straight line.

At time t s, the particle has displacement x m from a fixed origin O and is moving with velocity v ms^{-1}.

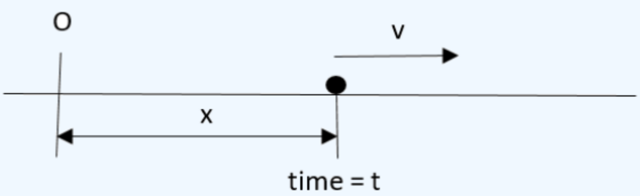
When $t = 1$, $x = -5$ and $v = 1$.

The acceleration a of the particle is given by

$$a = (16 - 6t) \text{ ms}^{-2},\ t \geq 0.$$

The particle passes through O with speed U when $t = T$, $T > 0$.

Find the possible values of U.



Acceleration is given by:

$$a = (16 - 6t)$$

Integrating to determine velocity:

$$v = \int a\ .dt \quad = \quad \int (16 - 6t)\,.dt$$

$$v = -3t^2 + 16t + c$$

When t = 1, v = 1, so:

$$1 = -3(1)^2 + 16(1) + c$$

$$c = 1 + 3 - 16 \quad = -12$$

The expression for velocity is: $v = -3t^2 + 16t - 12$

The displacement s of the particle is given by:

$$s = \int (-3t^2 + 16t - 12)\,.dt$$

$$= -t^3 + 8t^2 - 12t + c$$

When t = 1, x = -5,   so:

$$-5 = -(1)^3 + 8(1)^2 - 12(1) + c$$

$$c = -5 + 1 - 8 + 12 \quad = \quad 0$$

The equation for displacement is: $s = -t^3 + 8t^2 - 12t$

$$s = t(-t^2 + 8t - 12)$$

$$s = t(t - 6)(-t + 2)$$

The particle is at O when the displacement is zero.
This occurs when t = 2 seconds and when t = 6 seconds.

At t = 2: $v = -3(2)^2 + 16(2) - 12 \quad = \quad 8\,ms^{-1}$

At t = 6: $v = -3(6)^2 + 16(6) - 12 \quad = \quad -24\,ms^{-1}$