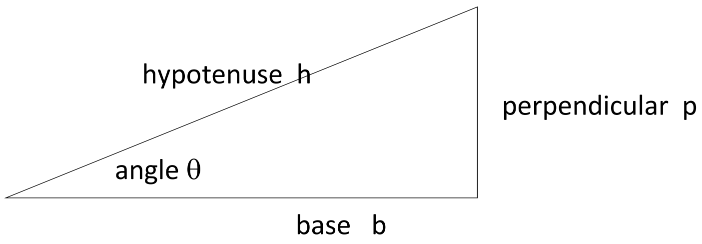
Graphs of trigonometric functions

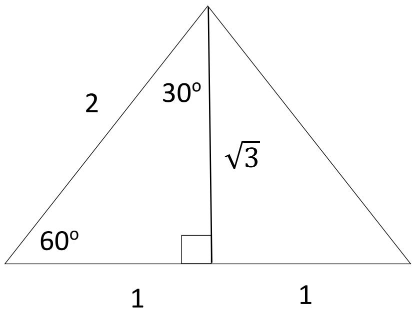
Learning points

The trigonometric functions sine, cosine and tangent are ratios of the lengths of the sides of a right angled triangle:



$$\sin\theta = \frac{p}{h} \qquad \cos\theta = \frac{b}{h} \qquad \tan\theta = \frac{p}{b}$$

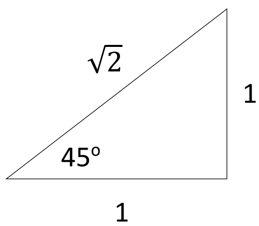
By considering an equilateral triangle with side length 2, divided into two halves, we can obtain exact values of the trigonometric functions for angles of 30° and 60°. The length of the vertical dividing line is given by Pythagoras' theorem: $\sqrt{2^2 - 1^2} = \sqrt{3}$:



$$\sin 30^0 = \frac{1}{2} \qquad \cos 30^0 = \frac{\sqrt{3}}{2} \qquad \tan 30^0 = \frac{1}{\sqrt{3}}$$

$$\sin 60^0 = \frac{\sqrt{3}}{2} \qquad \cos 60^0 = \frac{1}{2} \qquad \tan 60^0 = \sqrt{3}$$

In a similar way, the trigonometric functions for an angle of 45° can be found. A right angled triangle with two sides of length 1 unit is drawn:

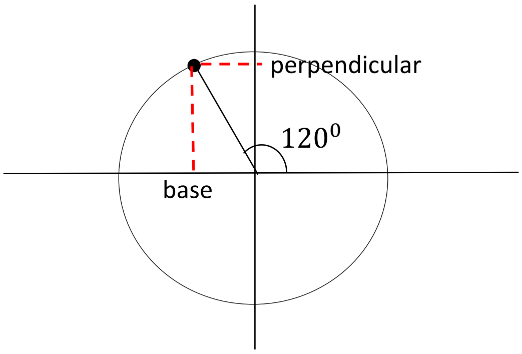


$$\sin 45^0 = \frac{1}{\sqrt{2}} \qquad \cos 45^0 = \frac{1}{\sqrt{2}} \qquad \tan 45^0 = 1$$

The angles found in this way are exact values. The square root terms can be evaluated to whatever degree of accuracy is required for a particular mathematical problem.

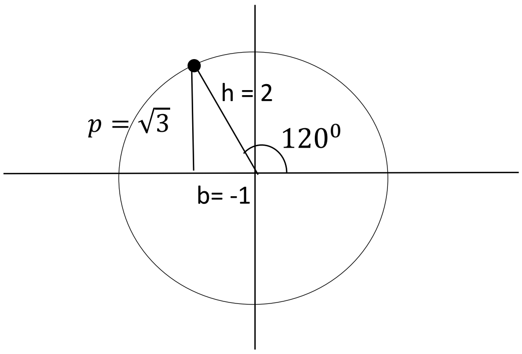
Trigonometric functions for angles greater than 90°

Sine, cosine and tangent functions for angles greater than 90° can be found by means of a circle with a line drawn from the centre to the point on the circumference at the required rotation angle. Angles are measured in an anticlockwise direction from the positive x-axis. For example, the angle 120°:



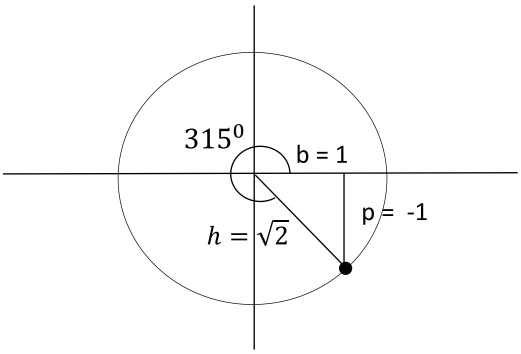
The trigonometric functions are again calculated as the ratios of the sides of a right angled triangle. The base is the x-value, the perpendicular is the y-value, and the hypotenuse is the circle radius.

For an angle of 120°, if we were to make the circle of radius 2 then the base would have a length of -1 and the perpendicular would be $\sqrt{3}$.



$$\sin 120^0 = \frac{p}{h} = \frac{\sqrt{3}}{2} \qquad \cos 120^0 = \frac{b}{h} = -\frac{1}{2} \qquad \tan 120^0 = \frac{p}{b} = -\sqrt{3}$$

The same method can be used for any angle. For 315°, a circle of radius $\sqrt{2}$ could be drawn. This would give a base value of 1 and a perpendicular value of - 1:



$$\sin 315^0 = \frac{p}{h} = -\frac{1}{\sqrt{2}} \qquad \cos 315^0 = \frac{b}{h} = \frac{1}{\sqrt{2}} \qquad \tan 315^0 = \frac{p}{b} = -1$$

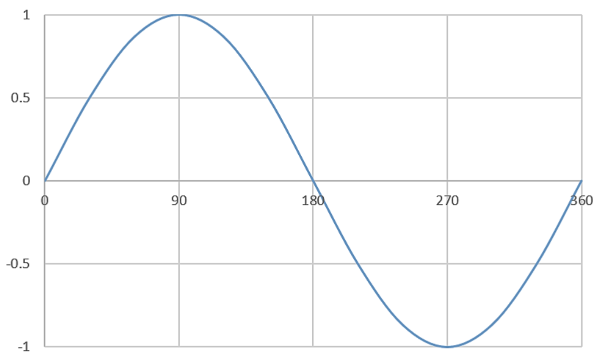
Notice that the triangle base and perpendicular may be either positive or negative, depending on the x-value and y-value at the required point on the circle circumference. However the hypotenuse, which is the radius of the circle, is always treated as positive.

Graphs of trigonometric functions

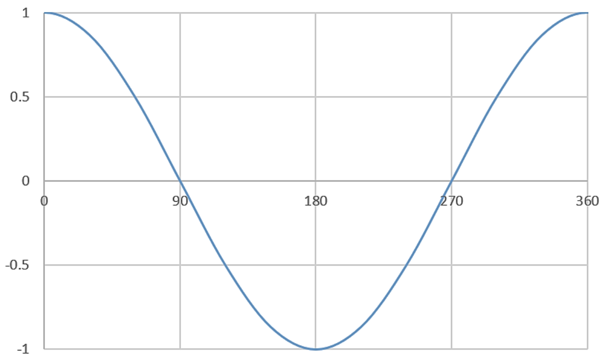
Graphs can be plotted using values calculated by the methods described above.

Sine and cosine both produce the same smooth curve with an amplitude of 2, meaning that the height difference between the peak and trough of the wave is 2 units. The wavelength, one complete wave pattern, is 360°. The curves have a phase difference of 90°, meaning that the peaks or troughs on the two curves occur at angles which are 90° apart.

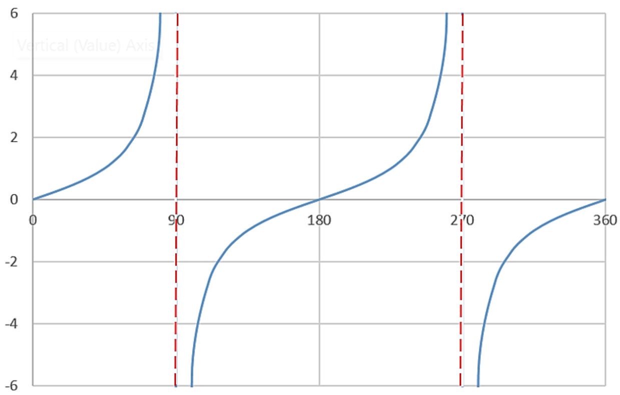
y = sin (x)



y = cos (x)



The tangent function produces a completely different wave pattern, made up from a series of unconnected segments. The tangent approaches assymptotes at 90°, extending infinitely in the positive and negative directions. The value of the tangent function is undefined at exactly 90°, and an error message appears if you attempt to find tan(90°) on a calculator.



The assymptote pattern is repeated at 180° intervals, with the next pair of assymptotes occurring at 270°.

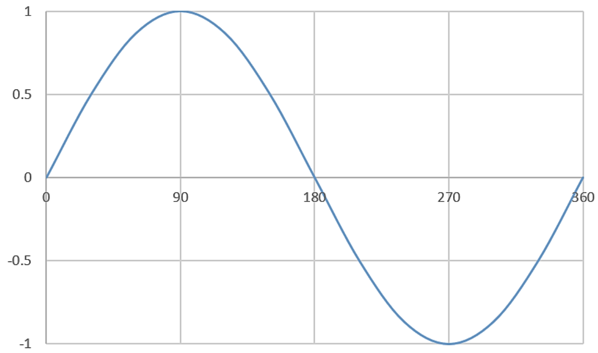
Graph transformations

It may be necessary to transform a basic trigonometric function to produce a mathematical model. For example:

> The water depth in a harbour is 2 metres at low tide and 5 metres at high tide. Low tide occurs on a particular day at 04:00 hours, and again at 16:00 hours.
>
> Tidal height can be modelled as a sine function. A function *f(d)* is required which can be used to produce a graph of water depth against time. The function will also allow water depth at a particular time to be calculated, or give the time(s) when the water has a particular depth.

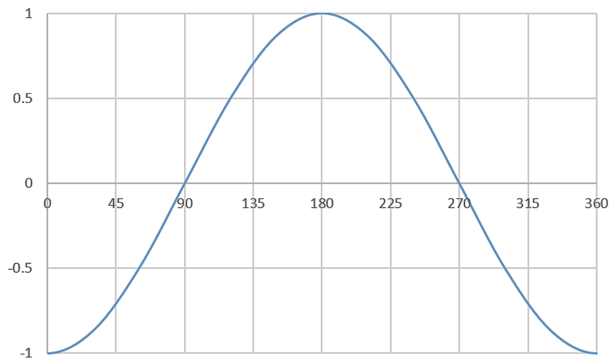
The model is developed in a series of stages:

- We begin with a basic sine curve: $y = \sin(x)$



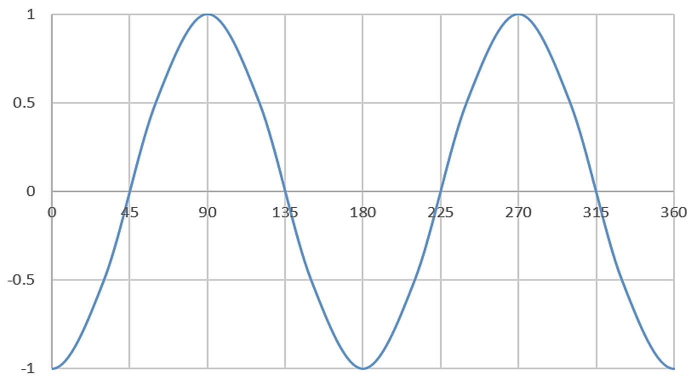
- It is convenient for the curve to start at a minimum point representing low tide. The graph can be translated in the positive direction by subtracting 90° from the angle:

$$y = \sin(x - 90)$$



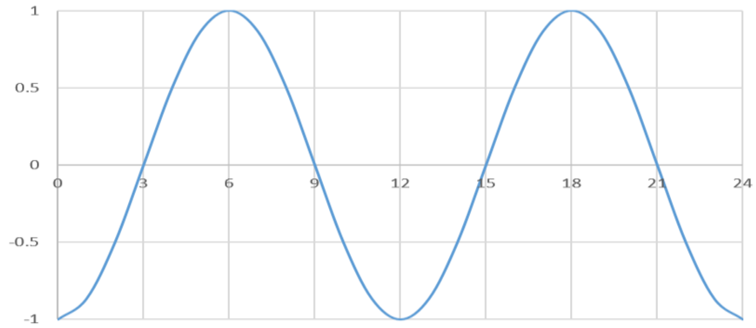
- It will be necessary to include two occurrences of low tide. The sine curve can be scaled by multiplying the angle x by 2:

$$y = \sin(2x - 90)$$



- The next step is to show times using the 24-hour clock. One wavelength of 180° represents the 12 hours between low tides, so 1 hour is represented by 15°. The scaling is done by increasing the multiplier for the angle:

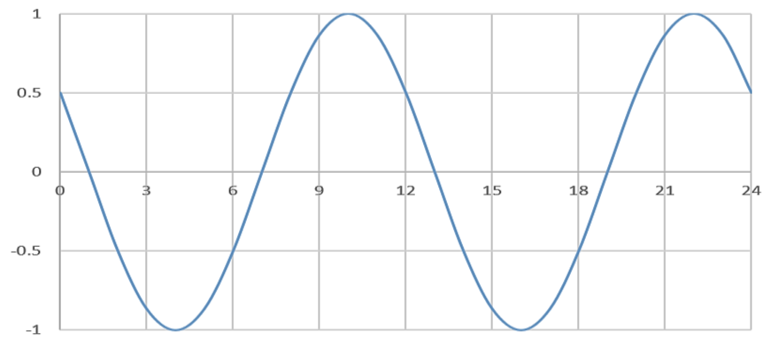
$$y = \sin(30t - 90)$$



Notice that the horizontal variable is now time t, running from 0 to 24 to represent one day. The time value has to be increased by a factor of 15 in order to convert from hours to degrees, before the sine value is found.

- One slight problem is that low tide occurs at 04:00 hours, not 00:00. This can be corrected by subtracting 4 from the hour t:

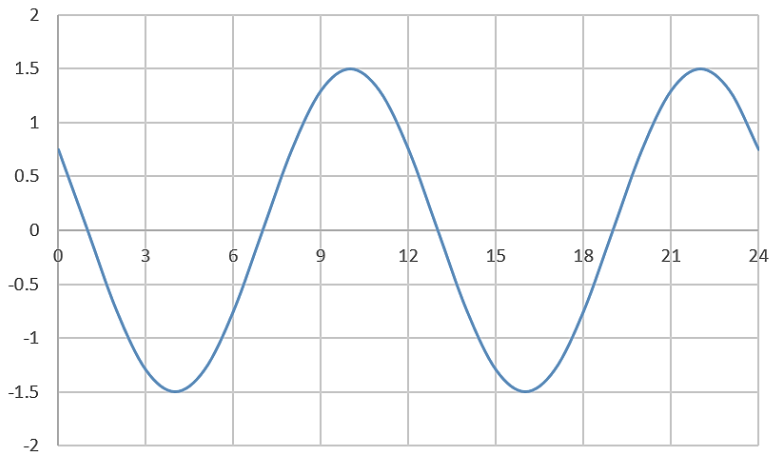
$$y = \sin(30(t - 4) - 90)$$



We now have a graph with high and low tides occurring at the correct times. The next step is to model the water depth d.

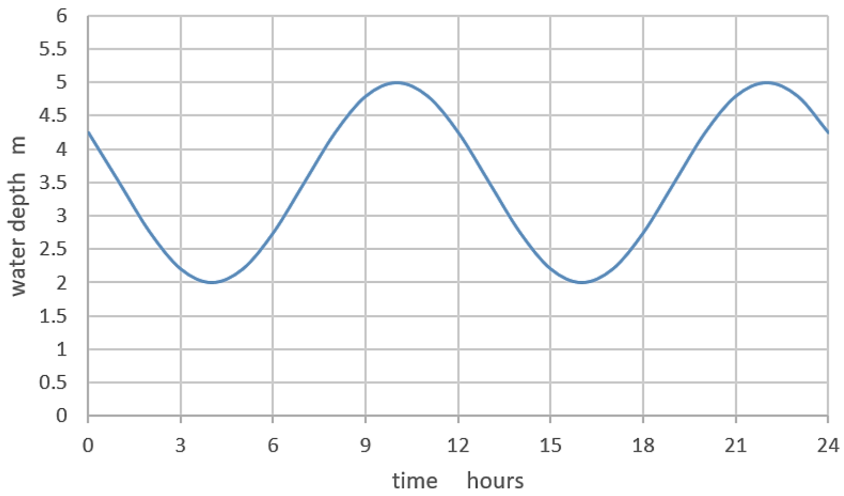
- The water depth varies between 2m at low tide and 5m at high tide, giving a tidal range of 3m. This can be represented by a sine curve of amplitude 1.5. We produce this amplitude by multiplying the output from the sine function:

$$y = 1.5\sin(30(t - 4) - 90)$$



- The final step is to move the graph upwards, so that the water depth at low tide is shown as 2m. This is done by adding 1.5 to the final output of the function so that the minimum occurs at 0, then a further 2 to increase the minimum value to 2:

$$y = 3.5 + 1.5\sin(30(t - 4) - 90)$$



This completes the model for the harbour water depth. Many sine or cosine models are constructed by following a similar sequence of steps. The function produced is often of the form:

$$y = a + b\sin(c(x) + d)$$

where:

a is the required vertical translation of the graph,
b is the vertical scaling of the curve, and represents half the height between the wave peak and trough,
c is a factor to convert the model horizontal variable, such as hours or days, into degrees,
d is an offset to ensure that peaks and troughs occur in the correct horizontal positions along the graph.

Radians

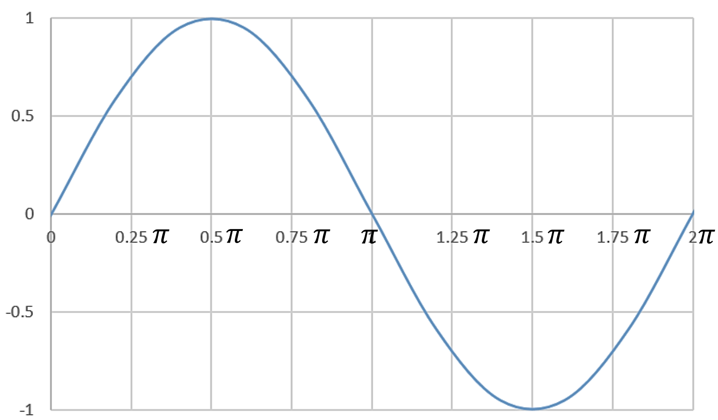
Angles in trigonometry are normally measured in degrees, but for some mathematical problems it is more convenient to use an alternative unit of measurement – radians.

Conversion between degrees and radians is straightforward:

$$360^o = 2\pi\ radians$$

Sine, cosine and tangent functions on calculators can generally be set to use either degrees or radians. The graphs of the trigonometric functions are the same shape for either unit of measurement. Using radians:

$$y = \sin(x)$$



Practical applications

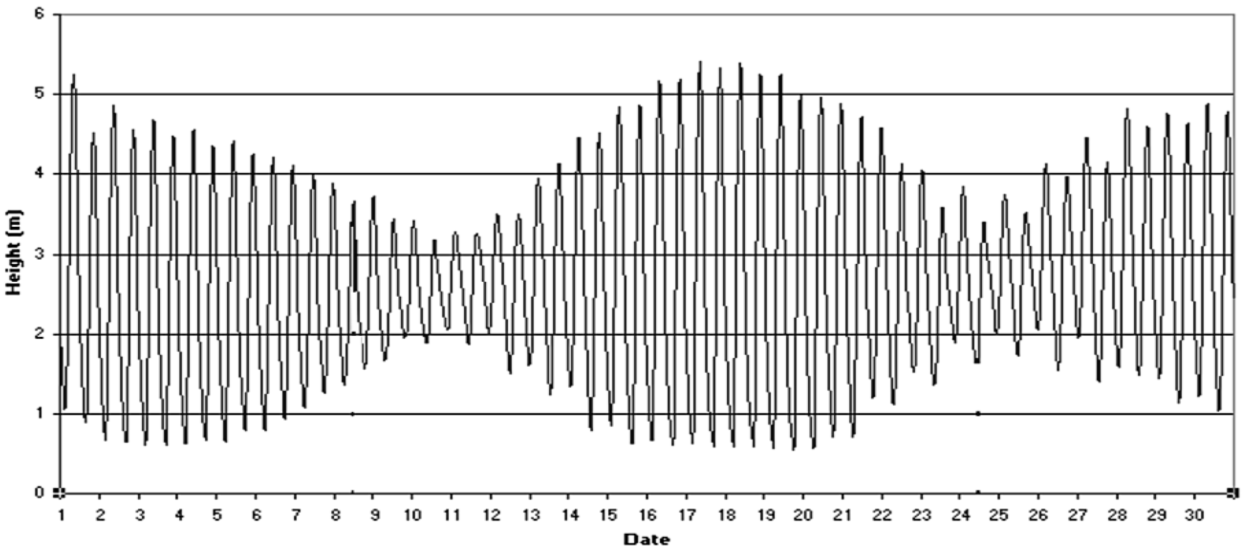
Trigonometric graphs have applications in many areas of science and technology. Some examples are given here:

Tides

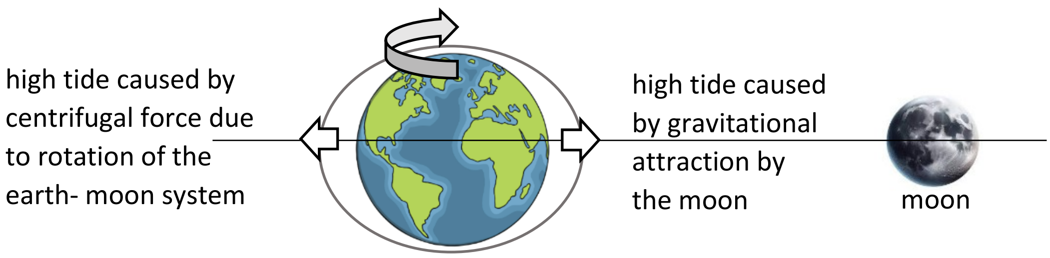
Trigonometric functions are used in the prediction of tidal heights.

Barmouth in North Wales has two high tides in each period of approximately 24 hours.

A tidal graph plotted for a typical month shows a complex wave pattern:



The two daily tides are caused by the moon. Gravitational influence on ocean waters by the moon causes a tidal bulge on the side of the earth nearest to the moon. However, the rotation of the earth-moon system also leads to a concentration of ocean waters on the opposite side of the earth. This region is furthest from the moon, so the gravitational effect is lowest and centrifugal force due to the earth's rotation increases the height of the water surface.

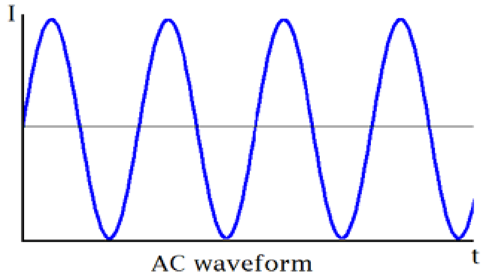


Superimposed on the daily tidal pattern is an approximately fortnightly variation between higher spring tides and lower neap tides. This is due to the gravitational effect of the sun, which at some times of the month acts in the same direction as the moon's gravitation, and at other times operates in a direction perpendicular to the moon.

A graph of predicted tidal heights can be produced by multiplying two cosine functions, one with a period of approximately 12 hours and another with a period of approximately 14 days.

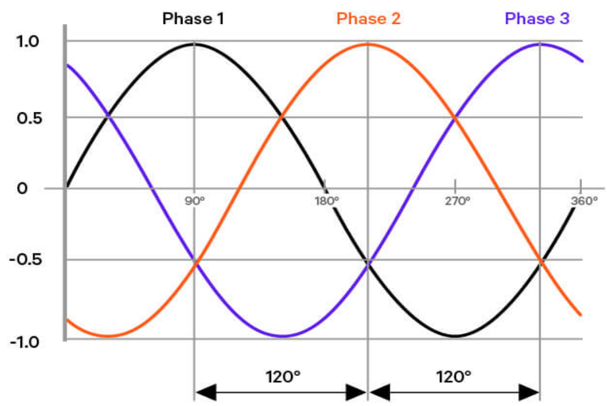
Electricity

The supply of electricity to domestic houses is an alternating current with a frequency of 60 cycles per second.



This works well for small appliances, but there is a problem in powering larger electric motors. During each cycle, the current varies between a maximum and zero. This means that the power supplied to the motor would be varying and unstable.

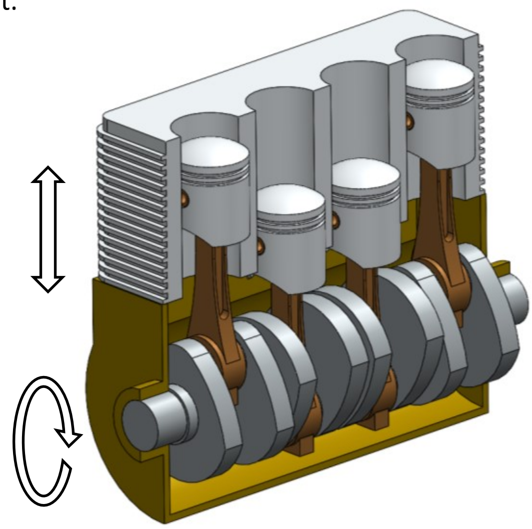
The problem is solved by providing three phase alternating current for industrial applications. The three waveforms have phase differences of 120°. At any instant, the total power supplied is near constant.



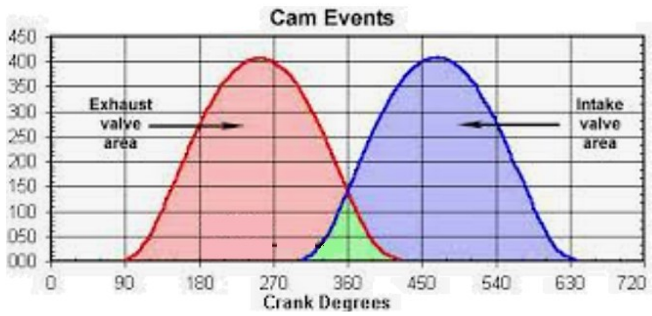
Machinery

Trigonometric graphs are useful in analysing the motion of machines, particularly where rotating and oscilating components are linked.

In a petrol or diesel engine, the linear movement of the pistons in the cylinders drives the rotational motion of the crank shaft.



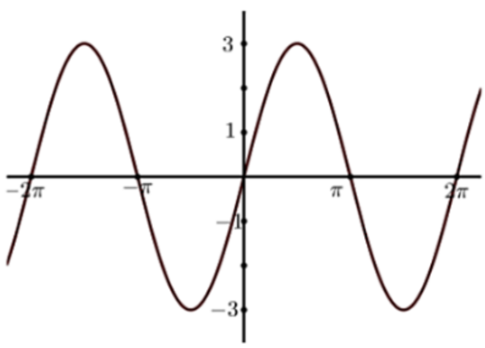
In turn, the rotational motion is transferred to a cam shaft which operates valves to let fuel and air into the cylinder at the start of the cycle, then allow the exhaust gases to escape at the end of the cycle.



Multiple choice questions

1.

Determine an equation of the form $y = A\cos(t)$ or $y = A\sin(t)$ for the given graph.

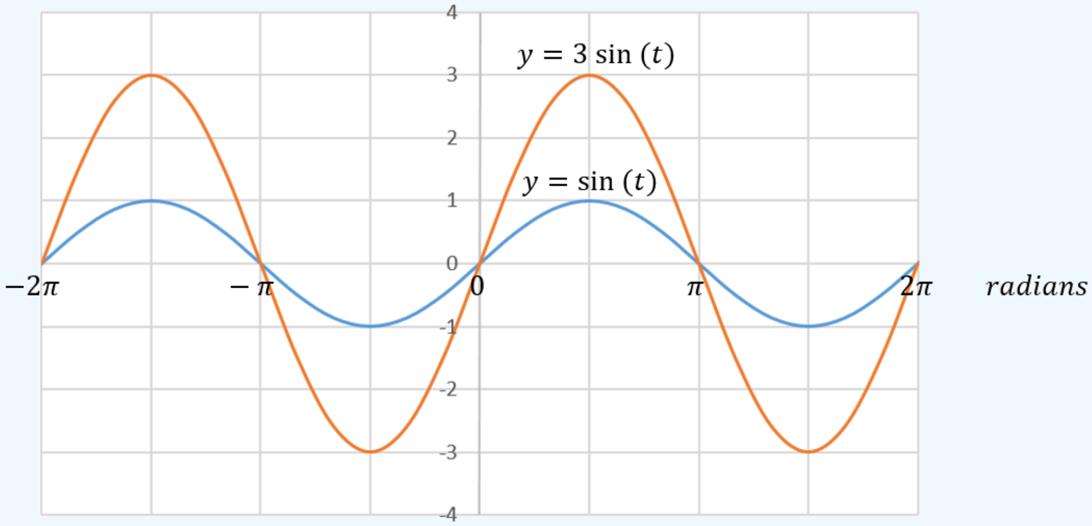


Select an answer

A: $y = \sin(3t)$ B: $y = 3\sin(t)$ C: $y = 3\cos(t)$ D:= $\cos(3t)$

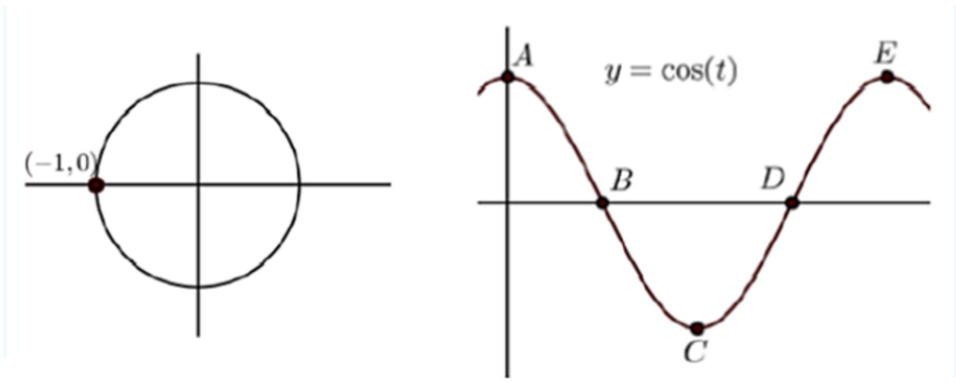
Correct answer: B

The graph is shown below in orange, with y = sin(t) shown in blue for comparison. Angles are given in radians, where $360^0 \equiv 2\pi\ radians$



2.

Match the point on the unit circle to a point on the graph of cos(t):

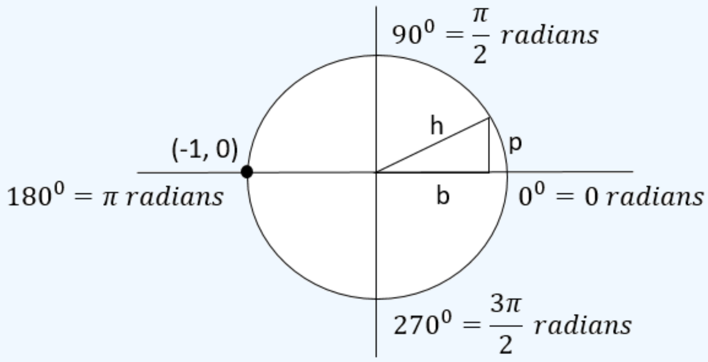


Select an answer

A: point B B: point D C: point C D: point E

Correct answer: C

Angles are measured in an anticlockwise direction from the positive x-axis. The cosine function is given by the ratio $\frac{b}{h}$ in a triangle drawn from the centre to the circumference, with the triangle base b taking the value on the horizontal axis, and the hypotenuse h having a value of 1 in any position:



At the point (-1,0), the angle is 180^0, or π radians. The value of the cosine function is:

$$\cos(\pi) = \frac{b}{h} = \frac{-1}{1} = -1$$

Point C has a value of -1

3.

Sketch, for $0 \le x \le 360°$, the graph of $y = \sin(x + 30°)$.
Use the graph to:
Solve, for $0 \le x < 360°$, the equation

$$\sin(x + 30°) = -\tfrac{1}{2}.$$

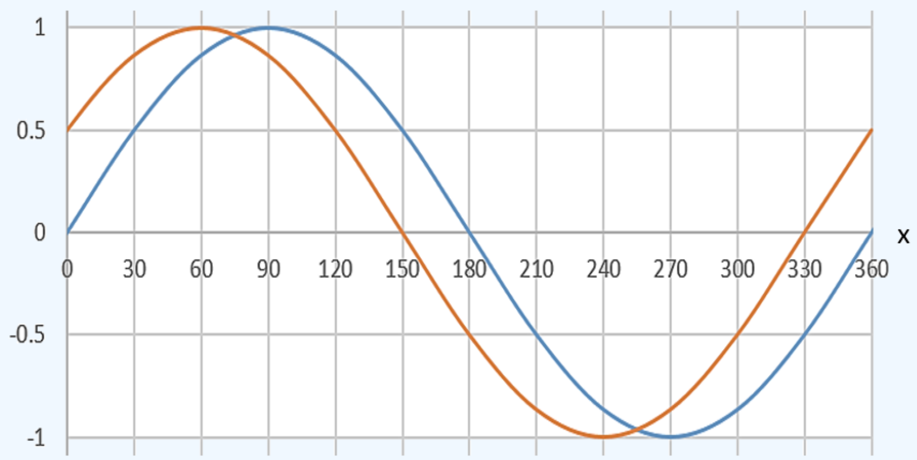
Select an answer

A: 120° and 330° B: 210° and 300°

C: 180° and 300° D: 150° and 330°

Correct answer: C

The graph of $y = sin(x + 30^o)$ is shown in orange below, with the graph of $y = sin(x)$ in blue for comparison.



The effect of adding 30^o to the angle before plotting its sine is to translate the graph in the negative direction by 30^o. For example, the value of 0.5 which represents sin(30^o) is plotted when x = 0.

It can be seen that the graph of $y = sin(x + 30^o)$ has values of $-\frac{1}{2}$ for angles x = 180° and 300°

4.

Given that

$$\cos(\theta - 20°) = \cos 60°$$

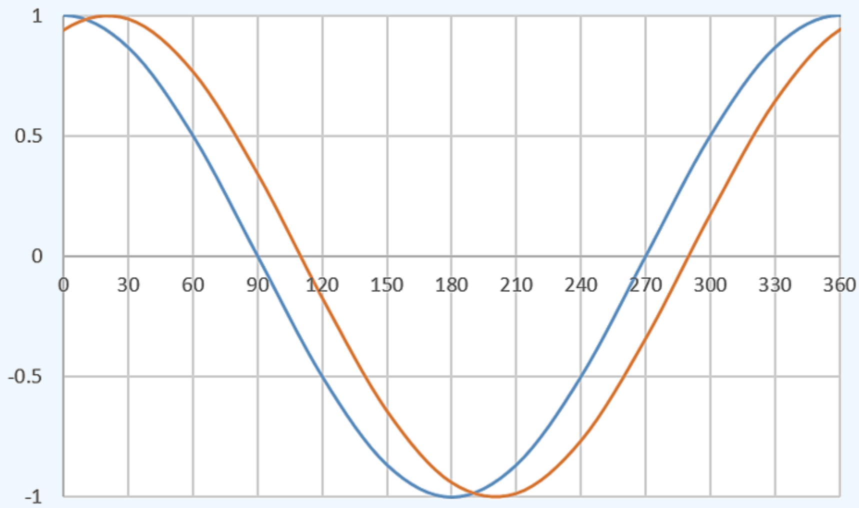
which one of the following is a possible value for θ?

Select an answer

A: 40° B: 140° C: 280° D: 320°

Correct answer: D

The graph of $y = cos(x - 20^o)$ is shown in orange below, with the graph of $y = cos(x)$ in blue for comparison.



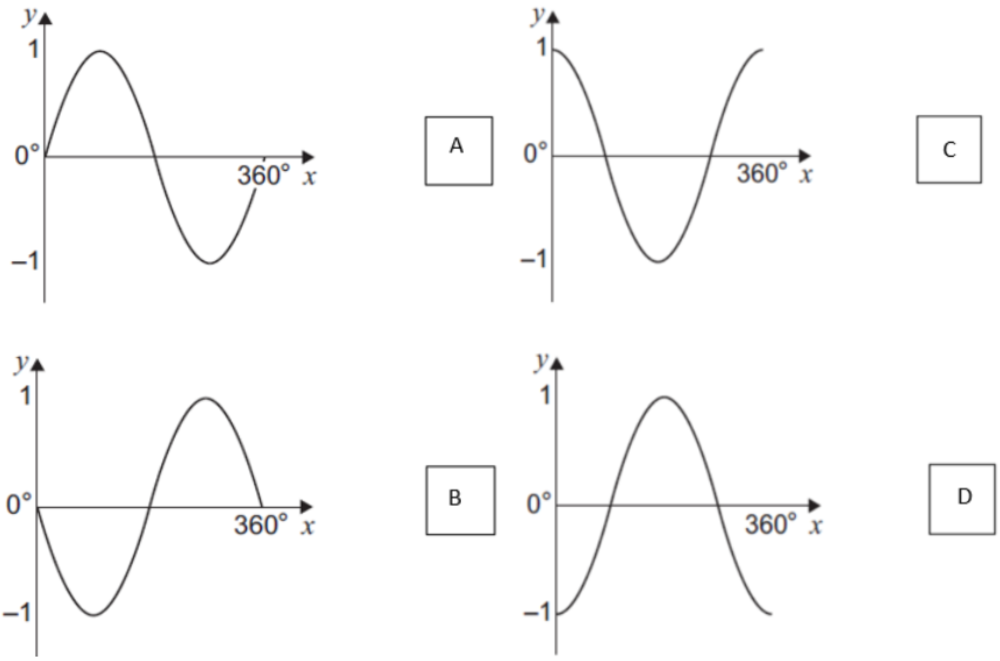
The effect of subtracting 20^o from the angle before plotting its cosine is to translate the graph in the positive direction by 20^o. For example, the value of 1.0 which represents cos(0^o) is plotted when x = 20.

$cos(60^0) = 0.5$, *so* possible solutions to the equation $cos(\theta - 20^o) = cos(60^0)$ are 80° and 320°

5.

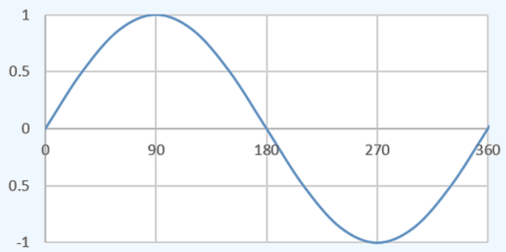
One of the diagrams below shows the graph of $y = \sin(x + 90°)$ for $0° \leq x \leq 360°$

Identify the correct graph.

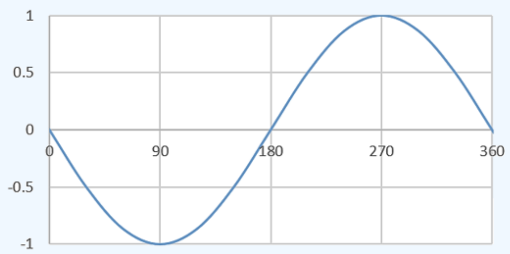


Correct answer: C

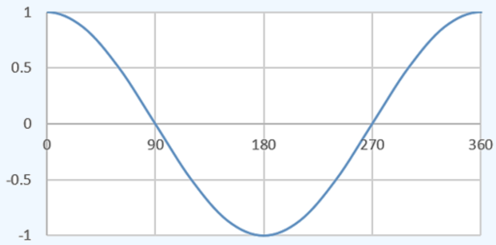
- Graph A is a basic sine function
  $y = \sin(x)$



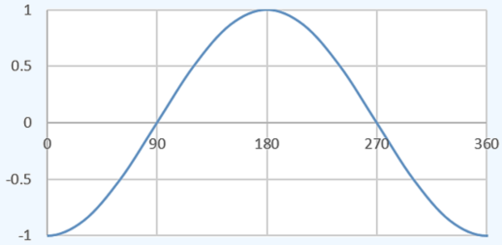
- Graph B is a negative sine function
  $y = -\sin(x)$



- Graph C is a sine function translated by 90° in the negative direction by adding 90° to the x value:
  $y = \sin(x + 90^0)$

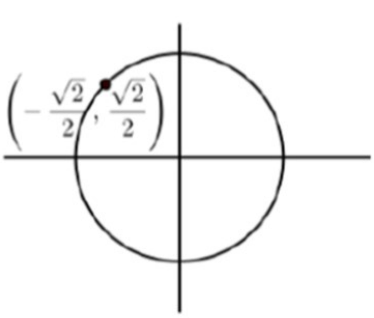


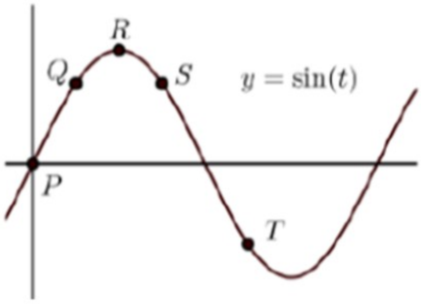
- Graph D is a negative sine function translated by 90° in the negative direction by adding 90° to the x value:
  $y = -\sin(x + 90^0)$



6.

Match the point on the unit circle to a point on the graph of sin(t):





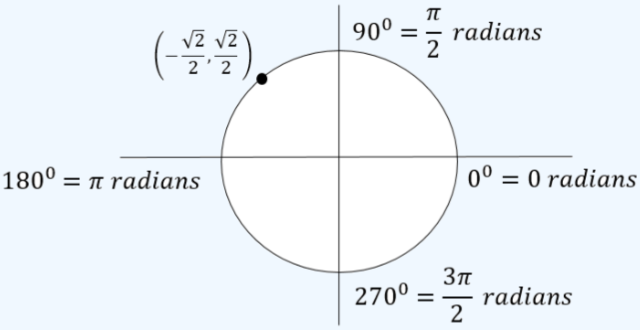
Select an answer

A: point Q B: point R C: point S D: point T

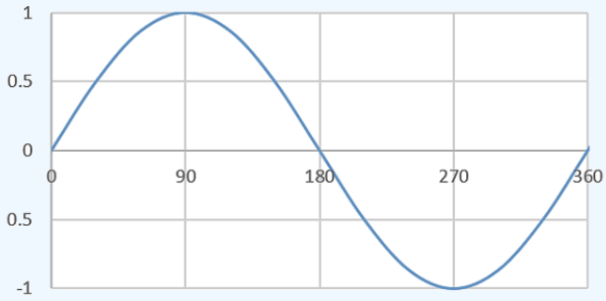
Correct answer: C

Angles are measured in an anticlockwise direction from the positive x-axis.

Tracing around the unit circle:



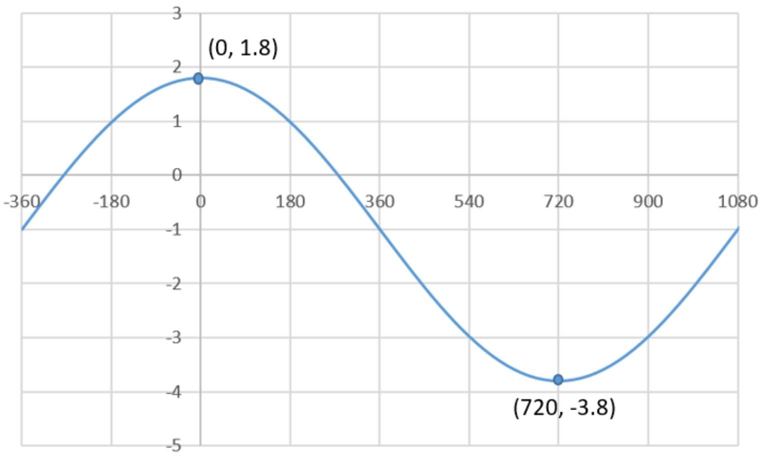
90^0 where sin(t) has a maximum has been passed, but 180^0 has not yet been reached where sin(t) is zero.



The marked point must therefore be point S on the graph.

7.

Determine an equation in the form: $y = A\sin\big(B(x + C)\big) + D$, where A, B, C and D are numerical values to be found, which produces the graph:



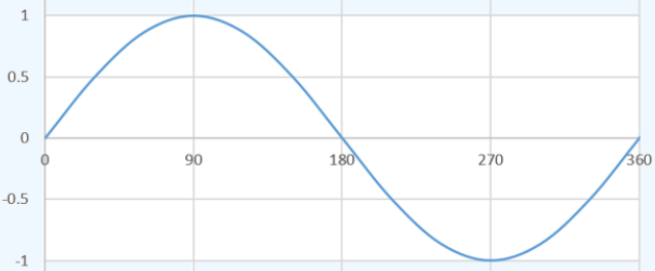
Select an answer

A: $y = 2.8\sin\left(\frac{1}{4}(x + 360)\right) - 1$ B: $y = 2.8\sin(4(x - 360)) - 1$

C: $y = 2.8\sin\left(\frac{1}{4}(x - 360)\right) + 1$ D: $y = 1.8\sin\left(\frac{1}{4}(x + 360)\right) - 3.8$

Correct answer: A

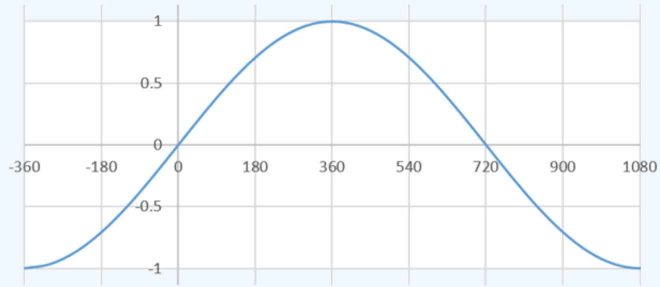
The graph shown above can be produced from a basic sine curve $y = \sin(x)$



by a series of transformations.

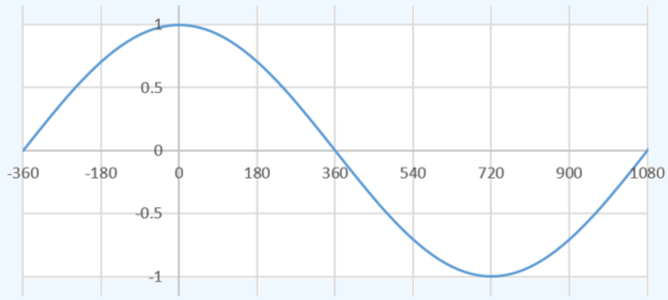
The wavelength of the sine curve in the question is 1440°, compared to the wavelength of a basic sine curve of 360°. The sine curve must be scaled by a factor of 4 in the horizontal direction.

A sine curve with the required wavelength is given by the equation: $y = \sin\left(\frac{1}{4}x\right)$



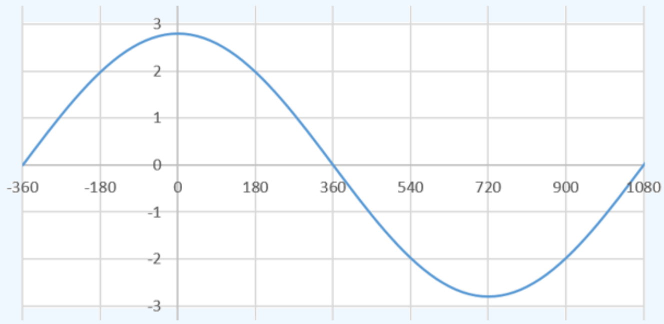
A range of x-values from -360° to 1080° is shown, as in the question graph.

The curve maximum should occur at 0°. This can be achieved by translating the curve by 360° in the negative direction. The required equation is: $y = \sin\left(\frac{1}{4}(x + 360)\right)$



The y-values in the question graph extend from a maximum of 1.8 to a minimum of -3.8, giving an amplitude of 2.8 which represents half this range.

The sine curve can be scaled vertically to give an amplitude of 2.8 by multiplying the output of the sine function: $y = 2.8\sin\left(\frac{1}{4}(x + 360)\right)$

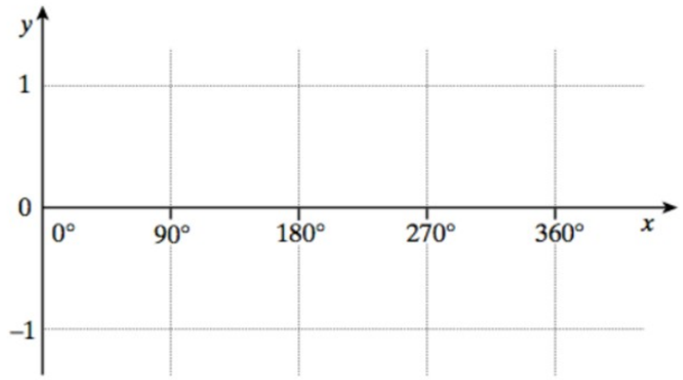


The graph in the question is not symmetrical about the horizontal axis, but is displaced downwards by 1 unit. This transformation can be achieved by subtracting 1 from the function output.

The final equation for the graph is: $y = 2.8\sin\left(\frac{1}{4}(x + 360)\right) - 1$

8.

Sketch the graph of y = cos x for $0 \leq x \leq 360$.



One solution of the equation cos x = 0.766 is x = 40°

Using your graph:

Find the other solution of this equation for $0 \leq x \leq 360$.

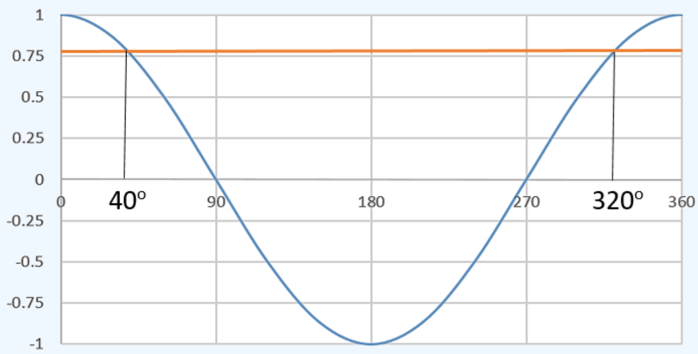
Select an answer

A: 140^o B: 220^o C: 310^o D: 320^o

Correct answer: D

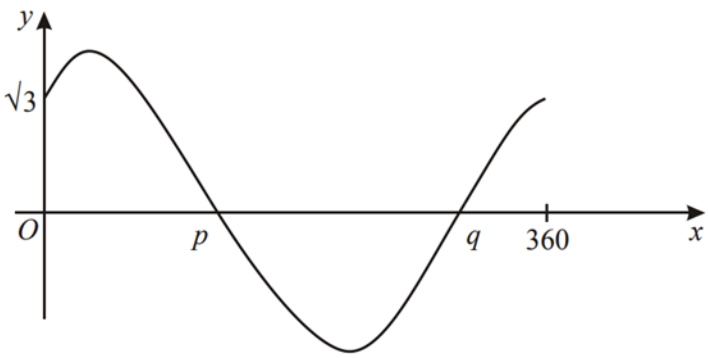
The cosine graph is shown below. A value of 0.766 occurs at 40°. From the symmetry of the graph, the same value of 0.766 must occur at

$$(360 - 40) \;=\; 320^o$$



Longer examination questions

1.



The diagram above shows the curve with equation $y = k \sin (x + 60)°$, $0 \leq x \leq 360$, where k is a constant.

The curve meets the y-axis at $(0, \sqrt{3})$ and passes through the points $(p, 0)$ and $(q, 0)$.

(a) Show that $k = 2$.

(1)

(b) Write down the value of p and the value of q.

The line $y = -1.6$ meets the curve at the points A and B.

(c) Find the x-coordinates of A and B, giving your answers to 1 decimal place.

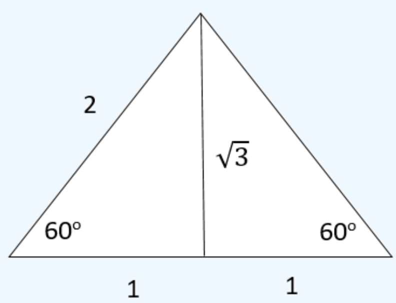
(5)

(Total 8 marks)

(a) The graph shows a sine curve which has been translated by 60° in the negative direction.

The translated curve cuts the vertical axis at $y = \sqrt{3}$, which represents a value of $k\ sin(60^o)$ where k is the amplitude of the sine curve.

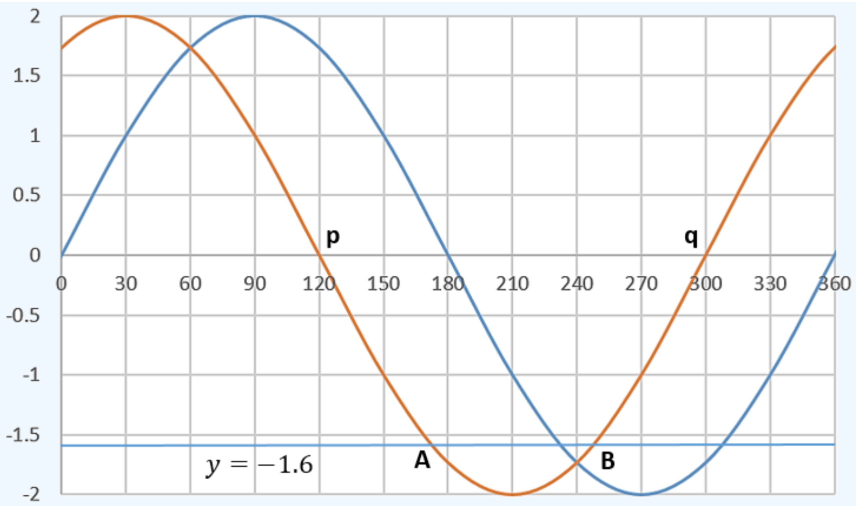
By considering an equilateral triangle with side length 2, divided into two halves:



By Pythagoras' theorem, the vertical dividing line has length $\sqrt{3}$. Therefore $sin(60^o) = \frac{\sqrt{3}}{2}$

Since $k\ sin(60^o) = \sqrt{3}$, then k = 2

(b) The basic sine function $y = 2\sin(x)$, shown in blue, crosses the x-axis at 180° and 360°. The function $y = 2\sin(x + 60)$, shown in orange, has been translated by 60°, so **p** = 120° and **q** = 300°.



Removing the scaling factor k, the line y = 0.8 would cut the basic sine curve at an angle of

$$\sin^{-1}(0.8) = 53.1^{0}$$

By the symmetry of the sine curve, the line y = - 0.8 would cut the basic sine curve at:

$$180 + 53.1 = 233.1^{o} \quad \text{and} \quad 360 - 53.1 = 306.9^{o}$$

The line cuts the translated curve at points which are 60° lower:

$$A = 173.1^{o} \qquad B = 246.9^{o}$$

2.

The curve C has equation $y = \cos\left(x + \frac{\pi}{4}\right)$, $0 \leq x \leq 2\pi$.

(a) Sketch C.

(2)

(b) Write down the exact coordinates of the points at which C meets the coordinate axes.

(3)

(c) Solve, for x in the interval $0 \leq x \leq 2\pi$,

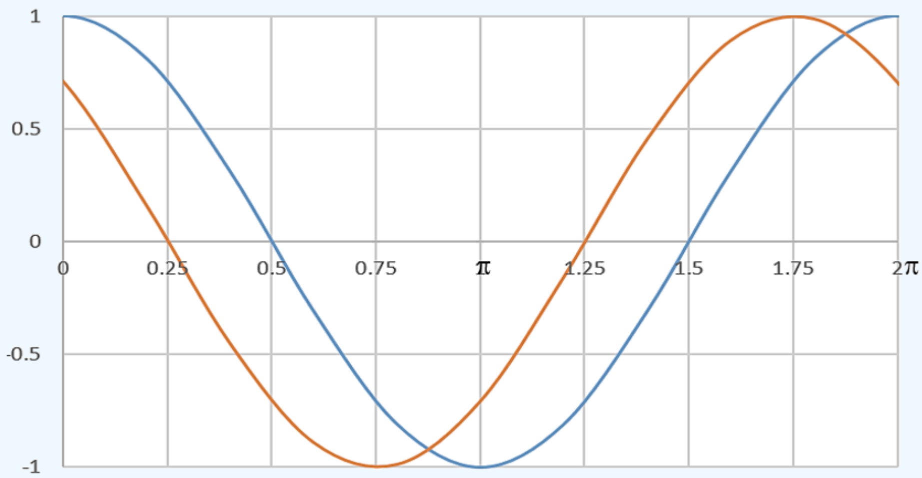
$$\cos\left(x + \frac{\pi}{4}\right) = 0.5,$$

giving your answers in terms of π

(4)

(Total 9 marks)

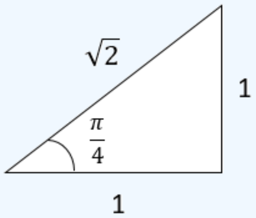
(a)



The graph of $y = cos\left(x + \frac{\pi}{4}\right)$ is shown in orange above, with the basic cosine curve $y = cos(x)$ shown in blue for comparison. Angles are measured in radians.

(b) The graph of $y = cos\left(x + \frac{\pi}{4}\right)$ has been translated by $\frac{\pi}{4}$ radians (45°) in the negative direction. The curve therefore cuts the x-axis at points which are $\frac{\pi}{4}$ radians less than the basic cosine curve. These points are $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{5\pi}{4}, 0\right)$

The graph cuts the y-axis at a value of $cos\left(\frac{\pi}{4}\right)$. The value can be deduced from a right angled triangle with angle of $\frac{\pi}{4}$ (45°) using Pythagoras' theorem:



$$cos\left(\frac{\pi}{4}\right) = \frac{base}{hypotenuse} = \frac{1}{\sqrt{2}}$$

Thus the graph of $y = cos\left(x + \frac{\pi}{4}\right)$ cuts the y-axis at the point $(0, \frac{1}{\sqrt{2}})$

(c) $cos(x) = 0.5$ when x = $\frac{\pi}{3}$ (60°) and when x = $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ (320°)

Thus $cos\left(x + \frac{\pi}{4}\right) = 0.5$ when x = $\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ and when x = $\left(\frac{5\pi}{3} - \frac{\pi}{4}\right)$

The solutions are therefore:

$$x = \frac{\pi}{12} \quad and \quad x = \frac{17\,\pi}{12}$$

3.

(a) Sketch, for $0 \leq x \leq 2\pi$, the graph of $y = \sin\left(x + \frac{\pi}{6}\right)$.

(2)

(b) Write down the exact coordinates of the points where the graph meets the coordinate axes.

(3)

(c) Solve, for $0 \leq x \leq 2\pi$, the equation

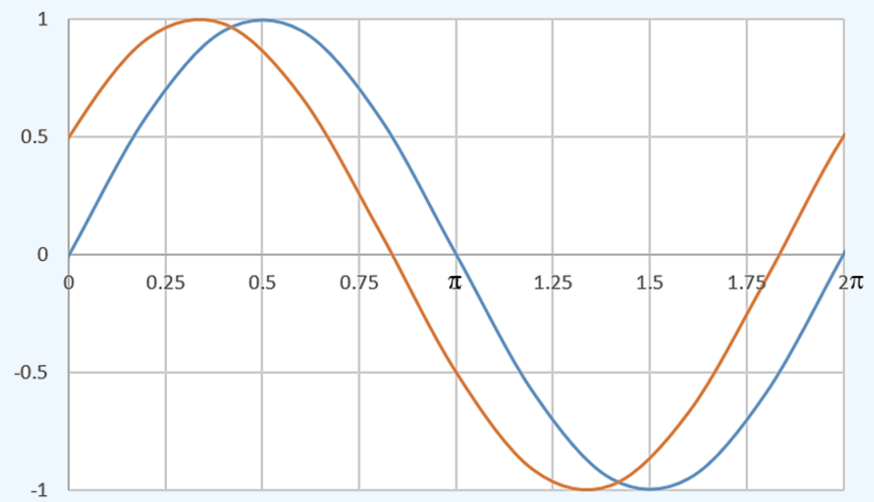
$$\sin\left(x + \frac{\pi}{6}\right) = 0.65,$$

giving your answers in radians to 2 decimal places

(5)

(Total 10 marks)

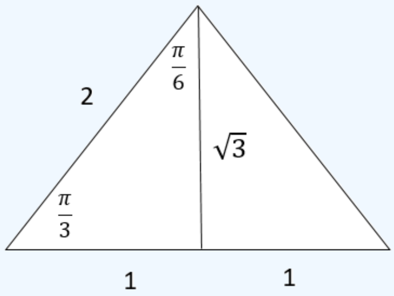
(a)



The graph of $y = sin\left(x + \frac{\pi}{6}\right)$ is shown in orange above, with the basic sine curve $y = sin(x)$ shown in blue for comparison. Angles are measured in radians.

(b) The graph of $y = sin\left(x + \frac{\pi}{6}\right)$ has been translated by $\frac{\pi}{6}$ radians (30°) in the negative direction. The curve therefore cuts the x-axis at points which are $\frac{\pi}{6}$ radians less than the basic sine curve. These points are $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$

The graph cuts the y-axis at a value of $sin\left(\frac{\pi}{6}\right)$, which can be found by considering an equilateral triangle with side length 2, divided into two halves:



By Pythagoras' theorem, $sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Thus the graph of $y = sin\left(x + \frac{\pi}{6}\right)$ cuts the y-axis at the point $\left(0, \frac{1}{2}\right)$

(c) $sin(x) = 0.65$ when x = 0.7076 radians and when x = $(\pi - 0.7076)$ radians

Thus $sin\left(x + \frac{\pi}{6}\right) = 0.65$ when x = $\left(0.7076 - \frac{\pi}{6}\right)$ and when x = $\left(\frac{5\pi}{6} - 0.7076\right)$

The solutions are therefore:

$$x = 0.18 \quad and \quad x = 1.91 \text{ radians}$$

---

4. On 18 March 2019 there were 12 hours of daylight in Inverness.

On 16 June 2019, 90 days later, there will be 18 hours of daylight in Inverness.

Jude decides to model the number of hours of daylight in Inverness, N, by the formula

$$N = A + B \sin t^\circ$$

where t is the number of days after 18 March 2019.

(a) (i) State the value that Jude should use for A.

[1 mark]

(a) (ii) State the value that Jude should use for B.

[1 mark]

(a) (iii) Using Jude's model, calculate the number of hours of daylight in Inverness on 15 May 2019, 58 days after 18 March 2019.

[1 mark]

(a) (iv) Using Jude's model, find how many days during 2019 will have at least 17.4 hours of daylight in Inverness.

[4 marks]

(a) (v) Explain why Jude's model will become inaccurate for 2020 and future years.

[1 mark]

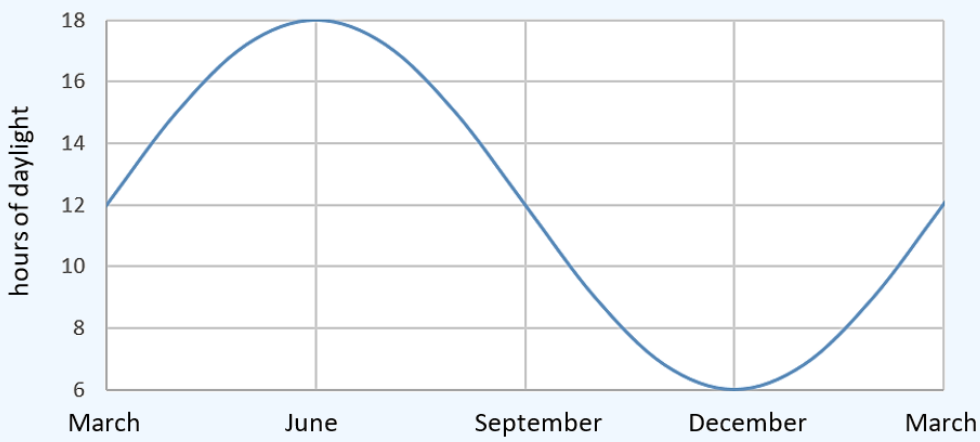
(b) Anisa decides to model the number of hours of daylight in Inverness with the formula

$$N = A + B \sin\left(\frac{360}{365}t\right)^\circ$$

Explain why Anisa's model is better than Jude's model.

[1 mark]

A sine curve: $N = A + B\, sin(t)$ is used to model the hours of daylight at different times of the year:



(a) (i) The mean (in spring and autumn) is 12 hours of daylight, so A = 12

(a) (ii) The amplitude (representing the extra hours of daylight in mid-summer) is 6, so B = 6

(a) (iii) 58 days after the start of the model, the number of hours of daylight will be:

$$12 + 6\sin(58) \quad = \quad 17.1 \text{ hours}$$

(a) (iv) When the number of hours of daylight is 17.4 then:

$$12 + 6\sin(t) = 17.4$$

$$6\, sin(t) = 5.4$$

$$sin(t) = 0.9$$

$$t = 64.16^o \quad or \quad t = (180 - 64.16) = \; 115.84^o$$

The number of days with at least 17.4 hours of daylight is given by:

$$115.84 - 64.16 = 51.68$$

To the nearest day, this is 52 days.

(a) (v) The model assumes that there are 360 days in a year, rather than 365 days (or 366 in a leap year).

(b) The new model keeps in sequence with the midsummer and midwinter solstices (maximum and minimum day length), and the spring and autumn equinoxes (equal 12 hour lengths of day and night).

---

5. The electricity supplied to residential houses is called alternating current (AC) because the current varies sinusoidally with time. The voltage which causes the current to flow also varies sinusoidally with time.

Both current and voltage alternate at 60 cycles per second, which can be approximated by a sine curve with a frequency of 18 miliseconds.

Let C be the current (in amperes), let V be the voltage (in volts), and let t be time (in miliseconds). The following list gives information that is known about C and V.

- The current C is a sinusoidal function of time with a frequency of 18 miliseconds, and it reaches its maximum of 5 amperes when t = 0 miliseconds. It reaches a negative maximum of -5 amperes when t = 9 miliseconds.
- The voltage V is a sinusoidal function of time with a frequency of 18 miliseconds.
- V "leads" the current in the sense that it reaches its maximum before the current reaches its maximum. The voltage V leads the current by 3 miliseconds, meaning that it reaches its maximum 3 miliseconds before the current reaches its maximum.
- The peak positive voltage is 240 volts. The peak negative voltage is -240 volts.

(a) Determine the equations of sinusoidal functions for both C and V.
Sketch the functions for C and V.

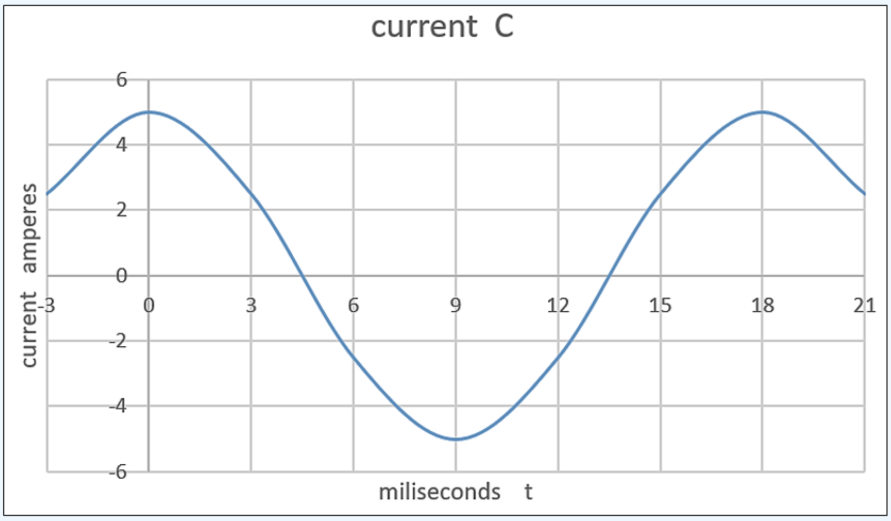
(b) Determine the voltage when the current is a maximum.

(c) Determine the current when the voltage is a minimum.

(d) Determine the current when the voltage is equal to zero.

(a) The time taken for one sine wave is 18 miliseconds. This value can be multiplied by 20 to represent 360°.

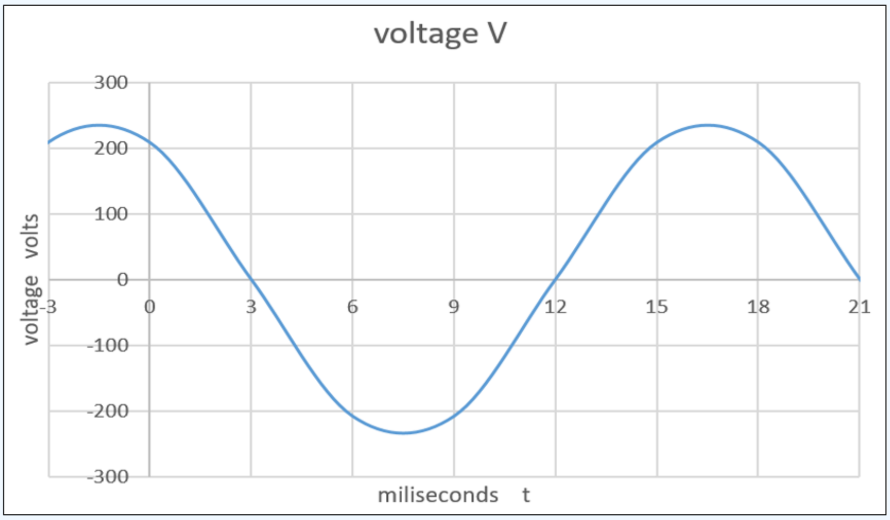
The amplitude of the graph for Current is 5 amperes. The peak of the curve must be translated by 4.5 miliseconds in the negative direction so that the maximum occurs at t = 0



An equation for current C is therefore:

$$C = 5\sin(20\,(t + 4.5))$$

The amplitude of the graph for Voltage is 240 volts. The peak of the curve must be translated by 6 miliseconds in the negative direction so that the maximum occurs at t = -3. This represents the 3 miliseconds by which voltage leads the current.



An equation for voltage V is:

$$V = 240\sin(20\,(t+6))$$

(b) The current is at a maximum when t = 0 miliseconds. At this time:

$$V = 240\sin(20\,(0+6))$$

$$V = 240\sin(120)$$

$$V = 207.8\ volts$$

(c) The voltage is at a minimum of -240V when t = 7.5 miliseconds. At this time:

$$C = 5\sin(20\,(7.5+4.5))$$

$$C = 5\sin(20\,(12))$$

$$C = 5\sin(240)$$

$$C = -4.33\ amps$$

(d) The voltage is zero at times t = 3 and t = 12 miliseconds. At these times:

$$C = 5\sin\big(20\,(7.5)\big)\ \ and\ \ C = \sin\big(20\,(16.5)\big)$$

$$C = 5\sin(150)\ \ and\ \ C = \sin(330)$$

$$C = \pm\,2.5\ amps$$

---

6. The depth of water, in metres, at a certain point in a harbour varies with the tide and can be modelled by a function of the form

$$f(t) = a + b\cos\left(c(t - d)\right)$$

where **t** Is the time in hours from the first high tide on a particular Saturday and **a**, **b**, **c** and **d** are constants. (Note: the cosine angle is expressed in radians.)

On that Saturday, the following were noted:

- The depth of the water in the harbour at high tide was 5.5 rn
- The depth of the water in the harbour at low tide was 1.7 m
- High tide occurred at 02:00 and again at 14:34.

(i) Use the Information you are given to add, as accurately as you can, labelled and scaled axes to the diagram below to show the graph of f over a portion of that Saturday.

The point P should represent the depth of the water in the harbour at high tide on that Saturday morning.

(ii) Determine the values of the constants **a**, **b**, and **c.** Write the equation for f(t) with the values of the constants included.

The constant **a** represents the vertical mid point of the sine curve:

$$a = \frac{1.7 + 5.5}{2} = 3.6$$

The constant **b** is half the amplitude of the wave, i.e. half the vertical distance between the peak and trough. This is given by:

$$b = (5.5 - 3.6) = \quad 1.9$$

The constant **c** is a multiplying factor which converts the period of 12.6 hours between high tides given by the variable **t** into a rotation angle of 2π:

$$c = \frac{2\pi}{12.6} \quad = \frac{\pi}{6.3}$$

The constant **d** translates the graph in the positive direction by 2 hours, as the first high tide occurs at 02:00. This can be done by simply subtracting 2 hours from the time value **t** :

$$d = 2$$

The complete equation is therefore:

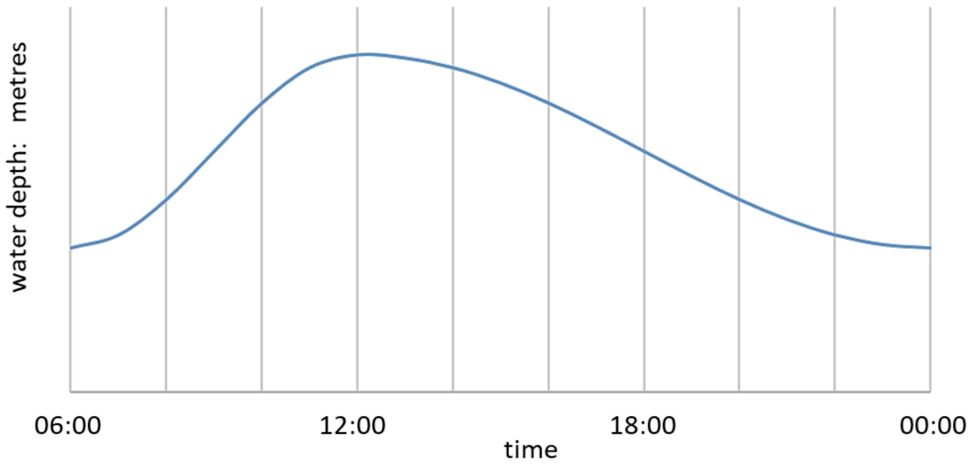
$$f(t) = 3.6 + 1.9 \cos\left(\frac{\pi}{6.3}(t-2)\right)$$

---

7. A large river flows through a town. The depth of water in the river is recorded at a monitoring point.

- On a particular day the river is flowing at its normal depth between midnight and 6am.
- A storm occurs at 6am, and the river level quickly rises until a peak flood level is reached at midday.
- The river level then gradually falls, returning to normal level at midnight.

The depth of water in the river is modelled by two sine functions, representing the rising and falling phases of the flood event:



For the rising phase between 6am and midday, the river level is modelled by the function:

$$d = 5 + 2\,sin\left(\frac{\pi}{6}\,(t+3)\right)$$

where **d** is water depth, **t** is the time in hours (using the 24 hour clock), and the angle of the sine function is given in radians.

For the falling phase from midday to midnight, the river level is modelled by the function:

$$d = 5 + 2\,sin\left(\frac{\pi}{12}\,(t-6)\right)$$

(a) Find the normal depth of the river before and after the flood.

(b) Find the maximum height reached by the river during the flood.

Emergency road closures are put in place whenever the river is at or above a depth of 6 metres.

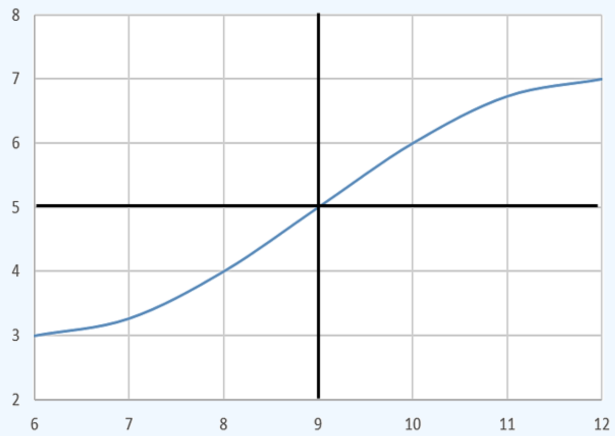
(c) Find the times between which the roads were closed during the flood.

The sine function for the rising phase of the flood is in the form:

$$d = a + b\ sin\big(c\ (t + d)\big)$$

where **a** is the mid line and **b** is the amplitude of the sine curve.

The function therefore has a mid line of 5 and an amplitude of 4 from the trough to the peak:



(a) The normal depth of the river is 3 metres.

(b) The maximum height reached is 7 metres.

(c) For the rising phase:

$$d = 5 + 2\ sin\left(\frac{\pi}{6}\ (t + 3)\right)$$

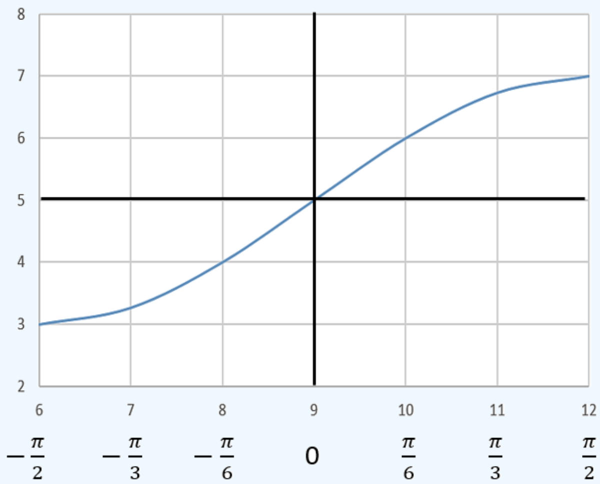
Letting the angle of the sine curve be θ, and the water depth 6 metres:

$$6 = 5 + 2\ sin(\theta)$$

$$\frac{6-5}{2} = sin(\theta)\ ,\ \ so\ \ \ sin(\theta) = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}\ radians,\ \ or\ 30^o$$

The time interval between 9am and noon is represented by sine angles from 0 to $\frac{\pi}{2}$ $, or\ 0\ to\ 90^o$

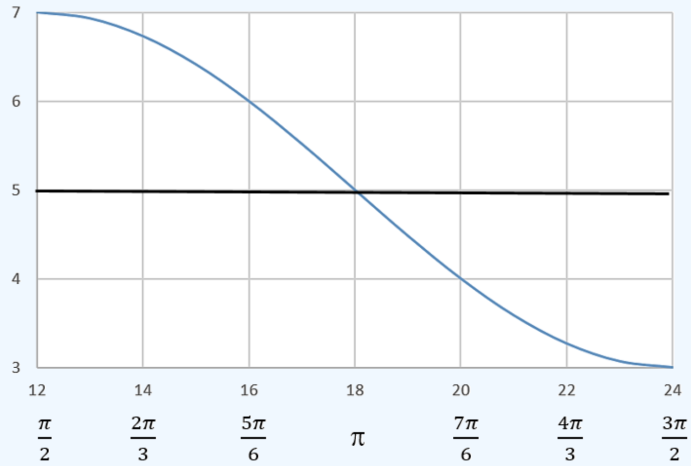


A sine angle of $\frac{\pi}{6}$ is reached at 10 am. This is the time that roads would be closed due to the flood.

For the falling phase, again: $$d = 5 + 2\,sin(\theta)$$

$$sin(\theta) = \frac{1}{2}$$

so $\theta = \frac{\pi}{6}$ radians, or $\frac{5\pi}{6}$ radians, which both have a sine value of 0.5



A sine angle of $\frac{5\pi}{6}$ is reached at 4 pm. This is the time that roads would be reopened after the flood.

---

8. A circular Ferris wheel is 120 metres in diameter and carries many cars. John enters a car at the bottom of the wheel and gets off 30 minutes later after going around once.

When a car is at the bottom of the wheel, it is 2 metres off the ground.

The height of a certain car on the Ferris wheel as it rotates can be modelled using a sine function.

(a) Sketch a graph of the function $h(t)$.

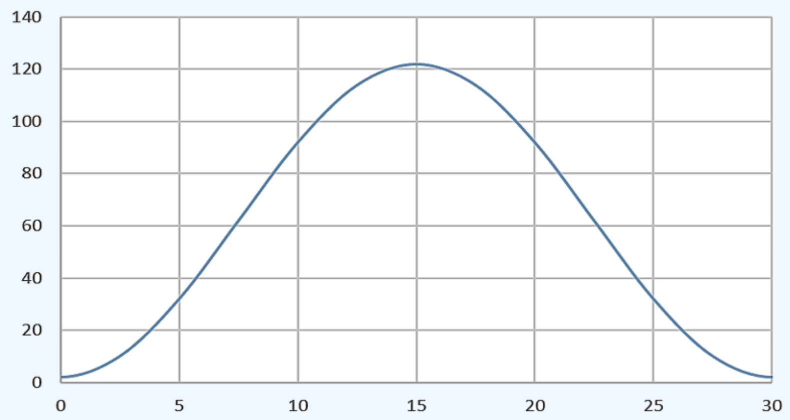
(b) Find the equation for the function $h(t)$.

(c) The ride suddenly stops 5 minutes and 30 seconds after John got on. How high is his car above the ground when this happens?

(d) How long does it take the car to first reach a height of 91 m above the ground?

(a) The car begins at a height of 2 metres and reaches a maximum height of 122 metres. One complete rotation takes 30 minutes.

Modelling the motion as a sine curve:



(b) The function will be of the form:

$$h(t) = a + b\ sin(ct - d)$$

where

- **a** is the height of the middle of the wheel and **b** is its radius,
- **c** is the multiplying factor needed to represent 30 minutes of time as an angle of 360°,
- **d** is the angular offset needed so that the sine curve minimum occurs at time zero.

$$h(t) = 62 + 60\ sin(12t - 90)$$

(c) The height at time t = 5.5 minutes is given by:

$$h(5.5) = 62 + 60\ sin(12(5.5) - 90)$$

$$= 62 + 60 \sin(-24)$$

$$= 37.6\ metres$$

(d) The time taken to reach a height of 91 metres is given by:

$$91 = 62 + 60\ sin(12t - 90)$$

$$\frac{29}{60} = sin(12t - 90)$$

$$sin^{-1}(12t - 90) = 0.483$$

$$12t - 90 = 28.90^0$$

$$12t = 118.90^0$$

$$t = \frac{118.90}{12} \quad = \quad 9.9\ minutes$$

The car reaches a height of 91 metres at a time of 9 minutes 54 seconds