

7 Shape and space

Mathematics is primarily involved with the identification of patterns in the behaviour of data and the use of these patterns in problem solving. This will often involve numbers, but in many important situations we are interested in the behaviour and properties of shapes. Working with shapes can increase spatial awareness in two- and three-dimensions and can develop mathematical skills in geometry and trigonometry. In addition, there may be opportunities for applying wider mathematical techniques, such as: ratios for converting scales, or algebra for transforming shapes. In this chapter, we examine various practical numeracy applications involving the mathematics of shape and space

Chapel conversion

Students in construction, design and technology, and computing can benefit from experience in the use of **architectural computer aided design** software. Various CAD systems suitable for students are available. Here, we describe the use of ArchiCAD (Graphisoft, 2008a, 2008b) in producing designs for the conversion of a disused chapel building into a family house.

Architectural CAD systems operate by assembling a collection of solid shapes, which are positioned in three-dimensional space to represent the building and its components. From the collection of shapes, the software is able to construct plans, elevations or three-dimensional models as required by the user. Particular components of the building, such as a roof, wall or floor, can be made transparent so that internal details of the structure are visible.

Complex components of the building can be constructed from simple shapes. This involves the application of logic operations on solid shapes. To demonstrate this process, consider two shapes, a cylinder and a cube, which have their centres at the same point in three-dimensional space:

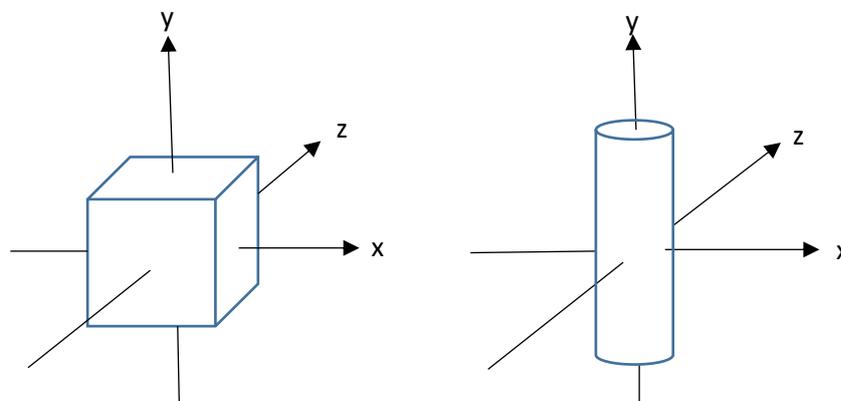
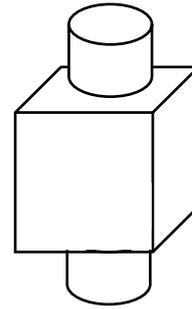


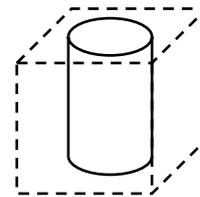
Figure 118: Solid shapes

The shapes can be combined in different ways:

- A logical OR operation produces a shape which includes all of the original cylinder and cube:

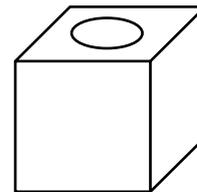


- A logical AND operation leaves only the central area in which the cylinder and cube overlap with one another:



Two different logical NOT operations are possible.

- In the first, we obtain all areas of the cube which are not also in the cylinder. This creates a hole through the cube:



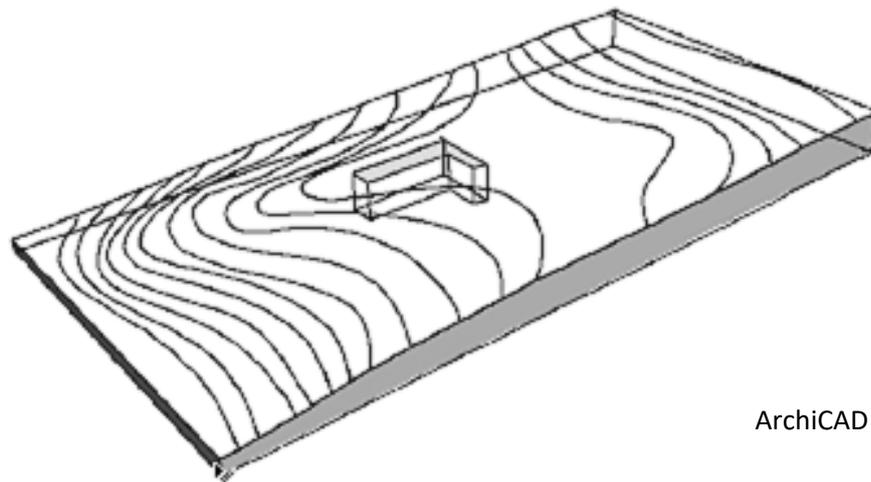
- In the second, we obtain all areas of the cylinder which are not also in the cube. This leaves just the two ends of the original cylinder:



Figure 119: Logic operations on the solid shapes

By a combination of these operations, complex building components can be constructed.

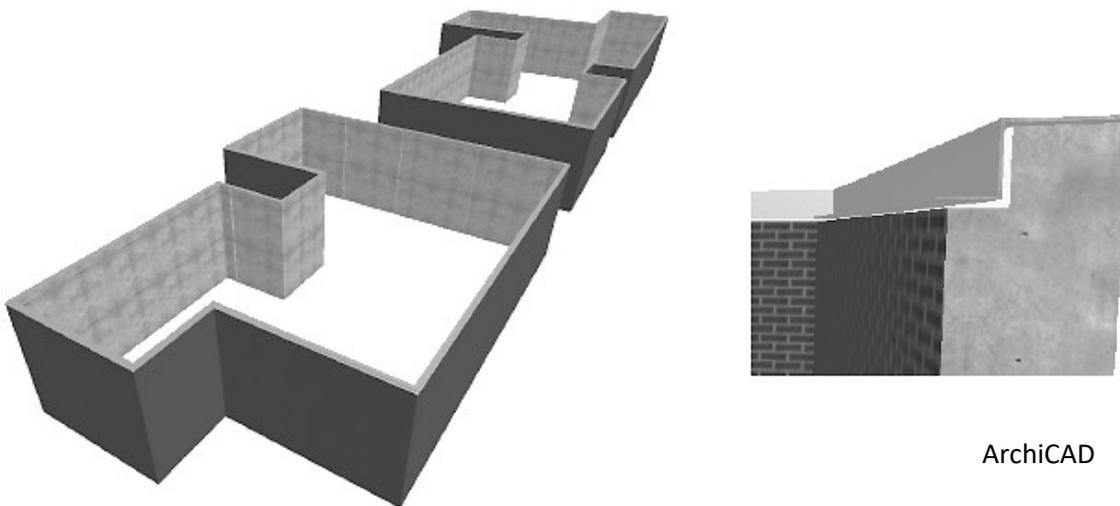
The design of a building begins with the layout of the site. The location may be flat and level, but often a building will be constructed on a sloping site. The software allows a series of spot heights to be entered from a topographic survey of the site. Software then interpolates between the known points to produce a contoured surface. The building foundation slab can then be positioned, as in figure 120.



ArchiCAD

Figure 120: Contoured model of the site, including the positioning of the building

Exterior and internal walls may be added to the foundation slab by entering their dimensions. The software allows the selection of a range of standard wall structures, but customised structures can also be build up from bricks, blocks, concrete, insulation or other materials.

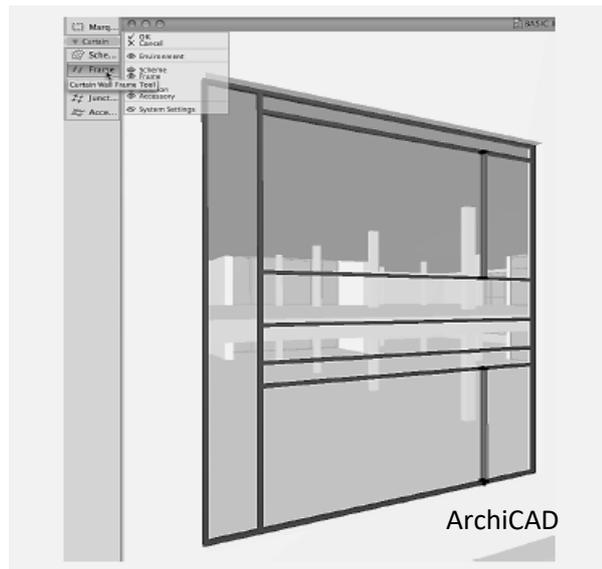


ArchiCAD

Figure 121: Layout of external walls, with details of wall structure

Standard doors and windows are provided, but again it is possible to construct custom doors and windows using panels, glazing bars and other components (figure 122).

Figure 122: Creating a custom window



A particularly important feature of architectural CAD systems is that complex building shapes can be represented, so that the creativity of the architect is not limited.

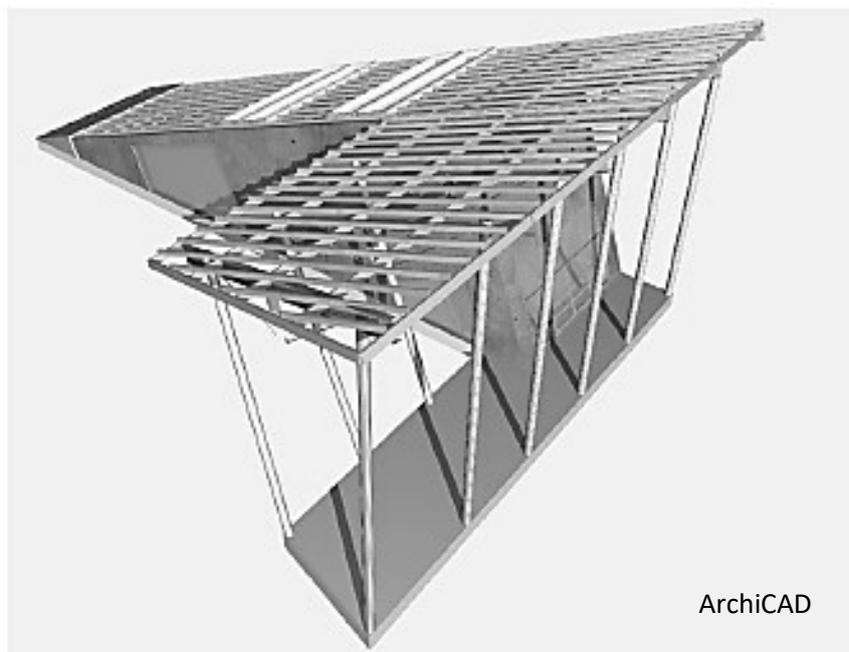


Figure 123: A non-standard building design, developed by geometrical and logical operations on solid shapes

A final stage in the design may be to add furniture and fittings to the interior, to assist the architect and their clients in visualising the completed building.

The exercise we describe here involves a building conversion project, in which the layout of a former chapel building was modified by the insertion of an internal floor and staircase, construction of internal walls, and addition of a new kitchen extension at the rear of the property.

Architect's drawings were provided, with dimensions either directly specified or available by scale measurement.

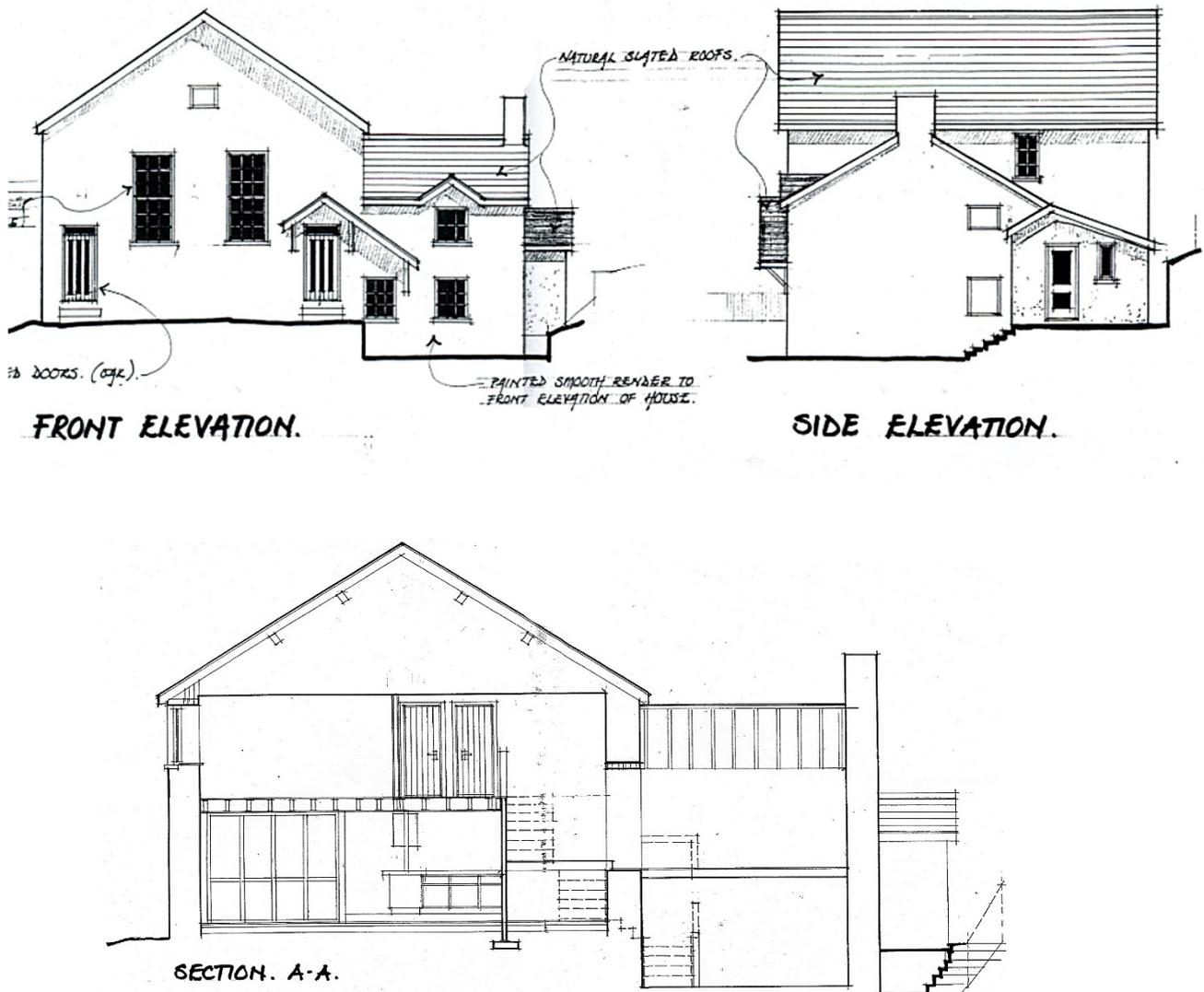


Figure 124: Architect's drawings for the chapel conversion

Using ArchiCAD software, the topography of the sloping site was created and the foundation slabs of the building were added at their appropriate elevations.

External and internal walls were then added, selecting appropriate construction materials.

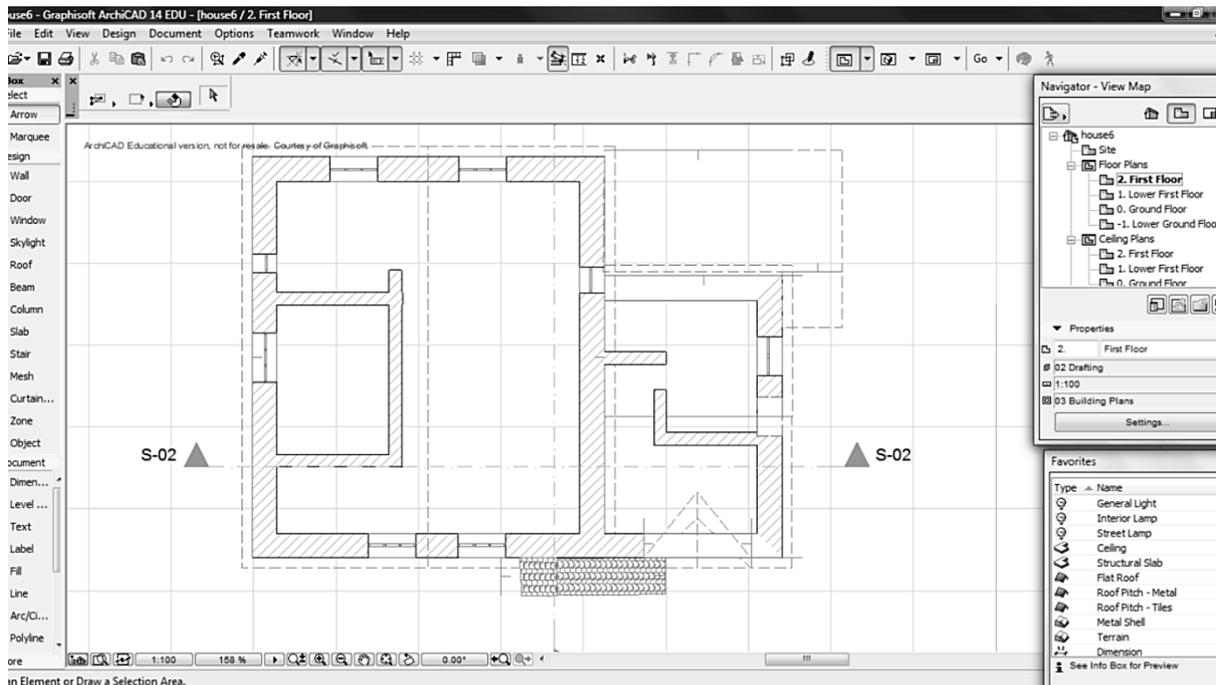


Figure 125: Input of external and internal walls

The software allows the display of the solid model at each stage. Switching to elevation views allows the shapes of walls to be modified, for example where the front wall of the chapel rises to the ridge line of the roof.

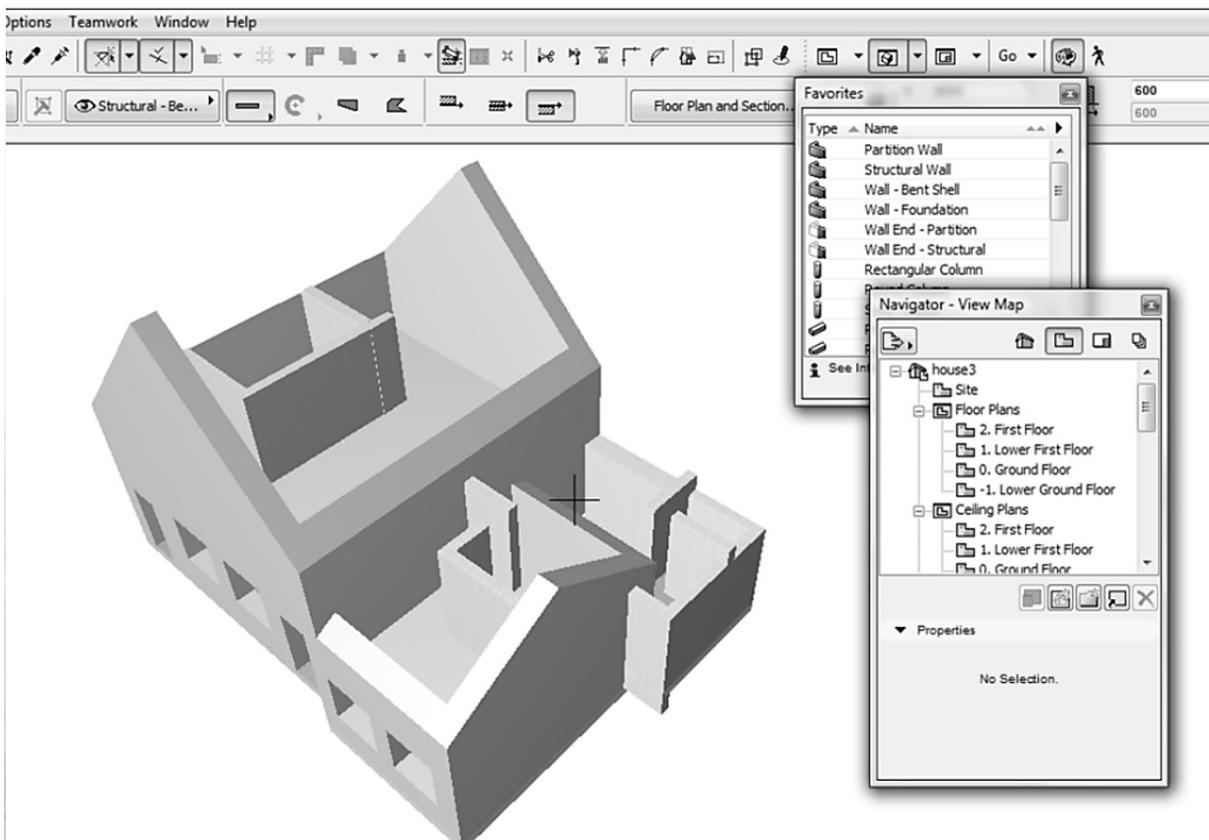


Figure 126: Solid model after adding external and internal walls

Roofs, windows and doors are added. Custom components were assembled according to the details shown on the architect's drawings.

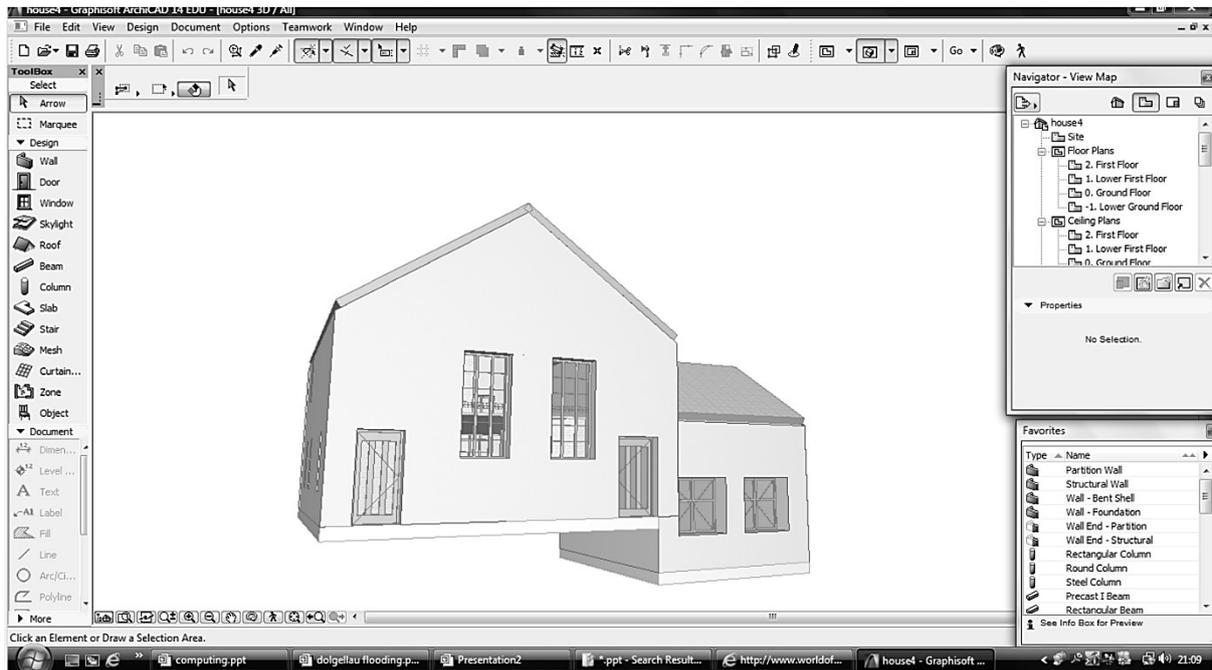


Figure 127: Roofs, windows and doors added to the model

The final stages of the model construction were to add external and internal features including porches, stairways, bathroom and kitchen fittings.

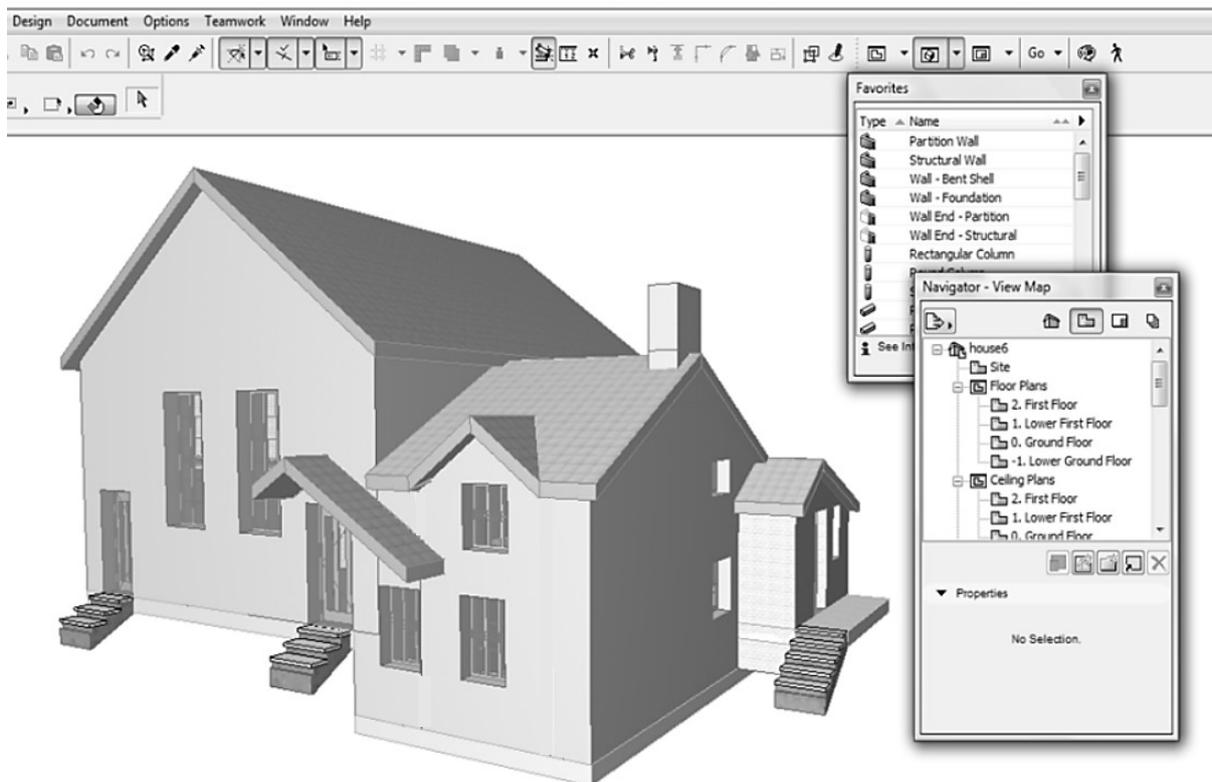


Figure 128: Addition of architectural details to the building

Curves in building construction

The components of the chapel described in the previous section were all created from flat surfaces, but it is not uncommon for curves to be used in building designs: for example in domes or arches. At different periods in history, and amongst different cultures, particular curved shapes have been favoured for the windows and doorways of castles, churches or other important buildings. The carpenters and masons who created these buildings used standard geometrical techniques to lay out the curves.

Amongst the more advanced practical skills taught to carpentry students are methods for constructing curved door and window frames of various designs. These designs have to be produced as a bench template for cutting the timber components. As an example, students can investigate the construction method for a Tudor arch:

- Two circles are drawn to create the required width for the arch
- A square is constructed, with the distance between the circle centres as one side
- The mid-point of the bottom edge CD of the square is used as a centre for constructing the upper arcs to complete the arch.

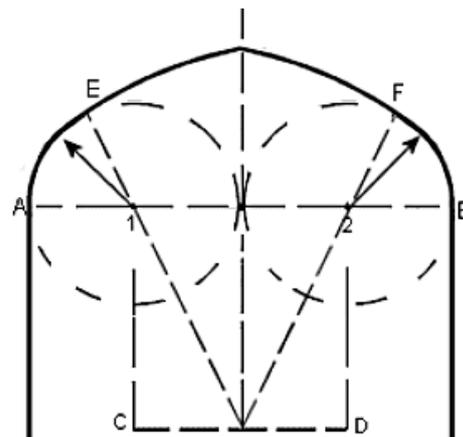


Figure 129: Geometrical construction for a Tudor arch



Figure 130: Ogee arch

Another interesting shape is the ogee arch, as illustrated in figure 130 above. The geometrical construction uses the following sequence:

- A semicircle is drawn with its centre at O to create the width AB of the arch.
- Using the diameter AB of the semicircle, two arcs are drawn from the points A and B so that they cross at point C.
- A rectangle ABQP is constructed, so that the upper edge passes through point C.
- Arcs are drawn from P and Q, passing through point C, which just touch the lower semicircle.

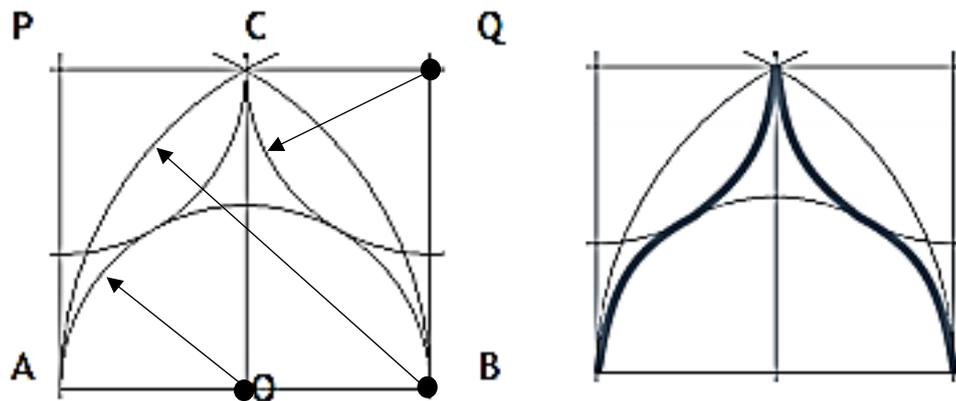
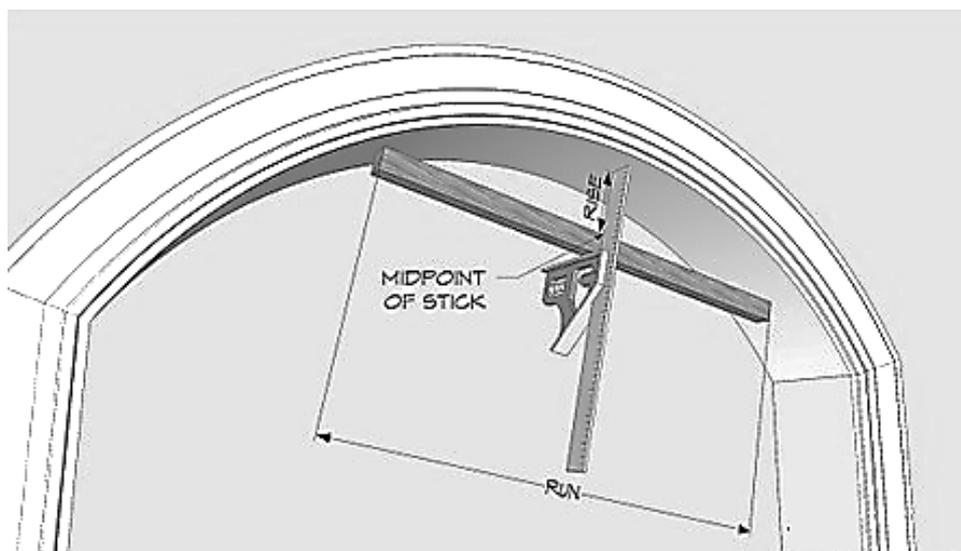


Figure 131: Geometrical construction for an ogee arch

Students can continue by investigating the geometry of other arch designs as a project.

A useful practical application of geometry is to determine the radius of an existing circular arch, for example when renovations are carried out.



www.thiscarpentry.com/2012/01/06/circular-based-arches-part-1/

Figure 132: Technique for determining the radius of a circular arch

- A piece of wood is measured, and its mid-point is marked. The length of the wood is recorded as a value called RUN. The wood is placed so that its two ends touch the arch.
- A square is positioned at the mid-point of the wood, and is used to measure the perpendicular distance from the top surface of the wood to the arch. This gives a value called RISE.

The radius of the arch can then be calculated using the formula:

$$radius = \frac{\left(\frac{run}{2}\right)^2 + rise^2}{2 \times rise}$$

This expression can be verified in the following way:

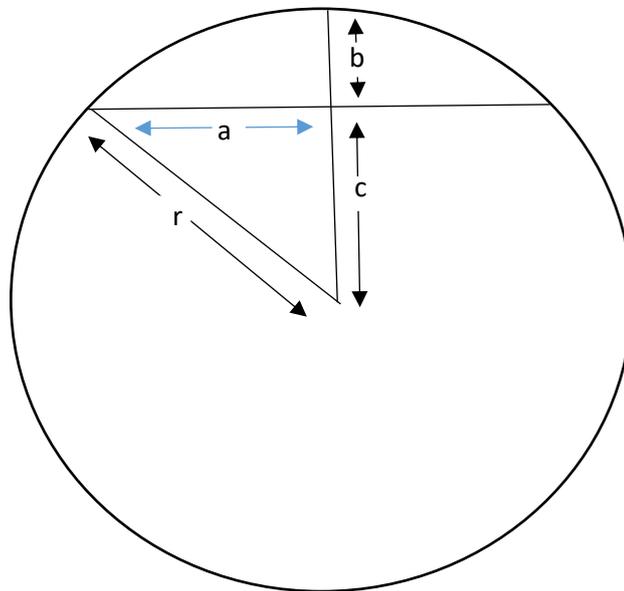


Figure 133: Derivation of the arch radius formula

Letting: the radius of the arch be **r**

the RUN of the piece of wood is **2a**

the RISE is **b**

By Pythagoras' theorem:

$$r^2 = a^2 + c^2$$

But: $r = b + c$

Rearranging: $c = r - b$

Substituting for c gives: $r^2 = a^2 + (r - b)^2$

Expanding the square term: $r^2 = a^2 + r^2 - 2rb + b^2$

Cancelling r^2

$$2rb = a^2 + b^2$$

Hence:

$$r = \frac{a^2 + b^2}{2b}$$

Substituting: **run** = $2a$, **rise** = b gives:

$$\text{radius} = \frac{\left(\frac{\text{run}}{2}\right)^2 + \text{rise}^2}{2 \times \text{rise}}$$

Another interesting application of curve geometry is in **brickwork**. Examples include the construction of circular walls for towers, and serpentine boundary walls made up from multiple curved panels. There may be practical requirements to calculate quantities of materials for curved brickwork, and to use geometrical techniques in laying out the curves on site.

Bricks may be laid using different bonds, depending on whether a single or double thickness is required for the wall (Hammett and Morton, 1991):

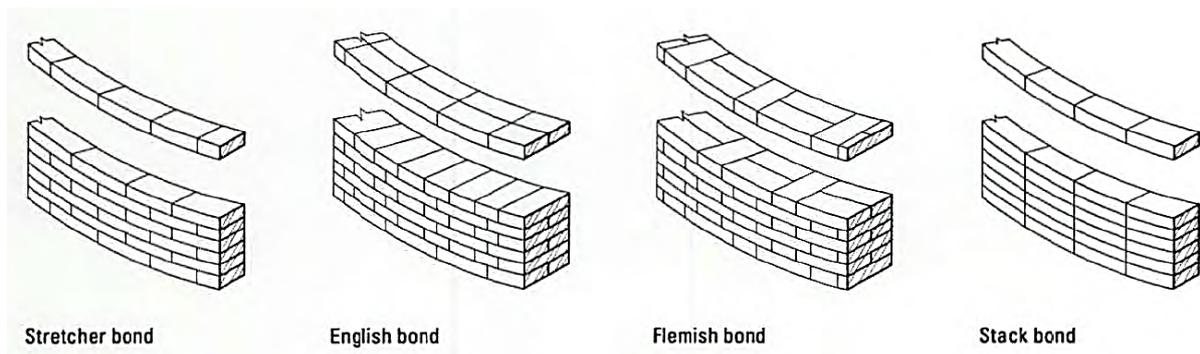


Figure 134: Brick bonds

Problems can arise if standard rectangular bricks are used for curved walls of small radius, and specialist bricks with curved faces are available:

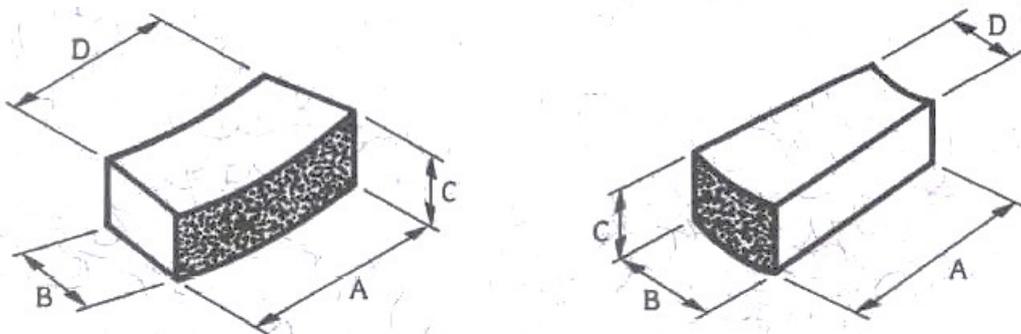


Figure 135: Specialist curved bricks: stretcher (left) and header (right)

Serpentine walls can be a particularly attractive feature of gardens. Reames (2012) discusses serpentine walls constructed in the early 1800's at the University of Virginia, USA. Interestingly, the curved pattern was chosen as a means of saving money. The curved wall has greater stability, so can be constructed with only a single thickness of brick, in comparison to a straight wall which would require a double thickness or the addition of strengthening pillars at intervals along the wall.



Figure 136: Serpentine wall

Laying out a serpentine wall requires the accurate positioning of the centres of the successive curves, on either side of the median line. A simple and practical technique used by builders at the University of Virginia was to lay out right angled triangles along the median line of the wall, using lengths of 3, 4 and 5 feet:

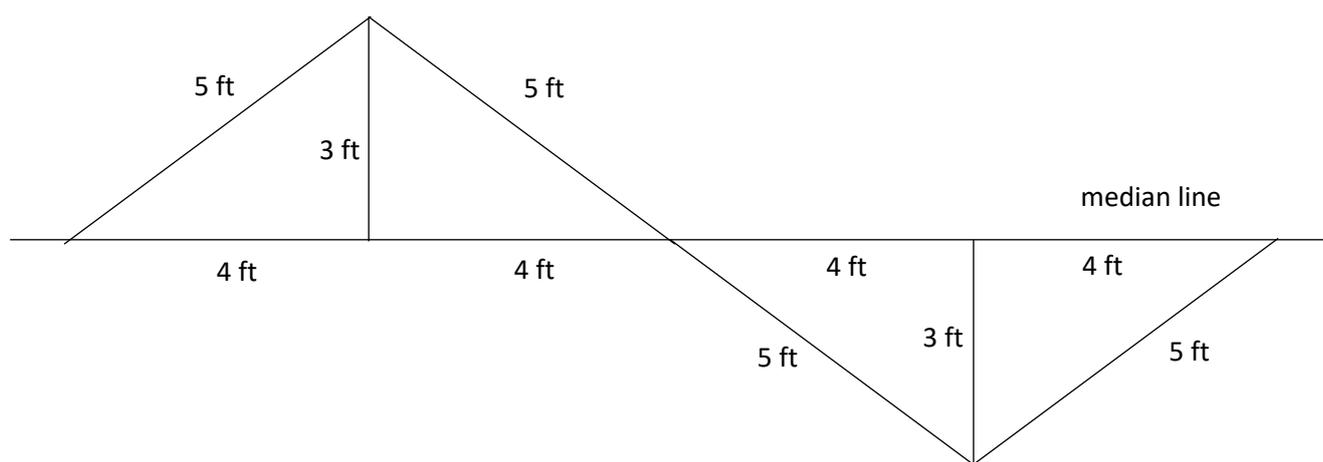


Figure 137: Geometrical construction for the serpentine wall

The arcs for the wall line can now be laid out by means of a string of length 5 feet, using the points P and Q as centres. The procedure can be repeated along the median line of the wall to produce the necessary number of curved bays.

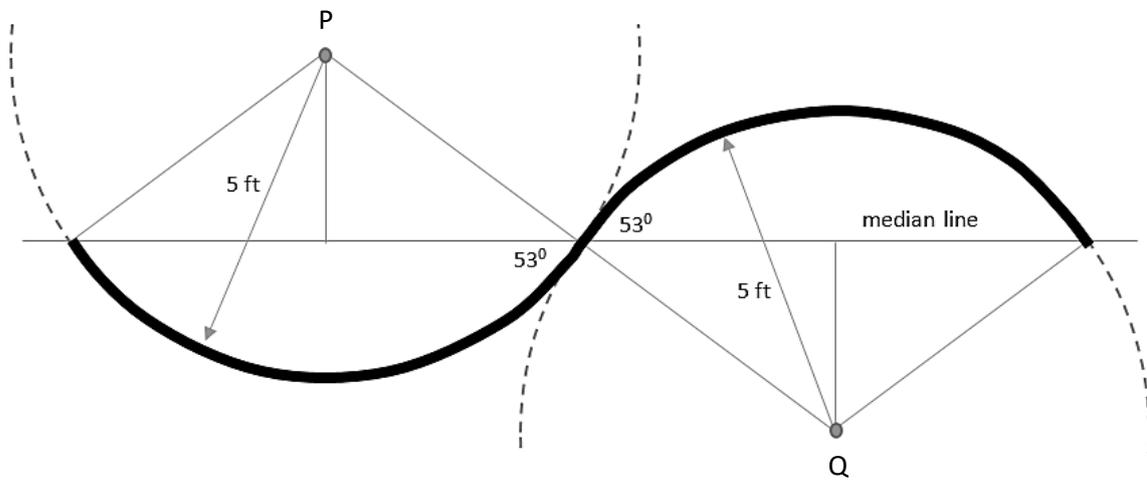


Figure 138: Laying out the line of the serpentine wall

It can be shown by trigonometry that each curve of the wall makes an angle of approximately 53° with the median line. Since the angles of approach are equal, the wall will appear as a continuous smooth curve.

For the general case, Hammett and Morton (1991) provide formulae linking the radius R , depth D , pitch L , centre line length I , thickness t , and angle of curvature Θ :

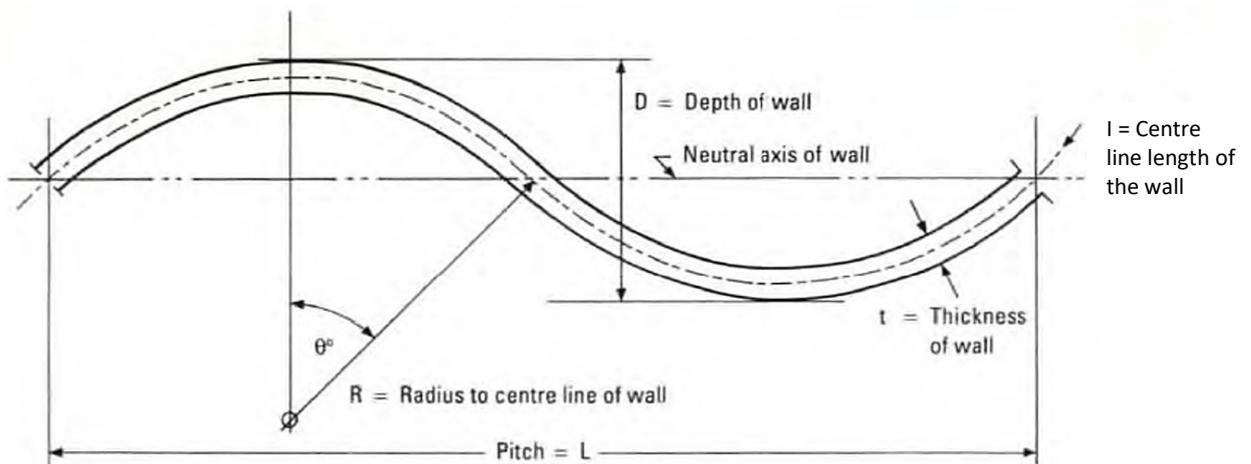


Figure 139: Parameters for a serpentine wall

$$L = 4 R \sin \Theta$$

$$I = 4 R \Theta, \text{ where } \Theta \text{ is measured in radians}$$

$$D = 2 R (1 - \cos \Theta) + t$$

It is useful for construction students to develop spreadsheets which allow the easy calculation of the parameters needed during the planning and laying out of curved brickwork. For a serpentine wall:

- The wall thickness would be decided by the construction method and brick bond
- The depth of the wall and the pitch of the curve would be known from the site plan.

The radius of curve can then be calculated. It is important that a correct radius is used, to ensure that the curved sections of the wall blend smoothly without any angular discontinuity.

Industrial archaeology site recording



In various courses it may be necessary to make accurate surveys of areas of land: for example, measuring the shape and slope profile of a construction site, or measuring the cross profile of a river channel in a geography project. In this section, we examine surveying techniques using simple surveying equipment.

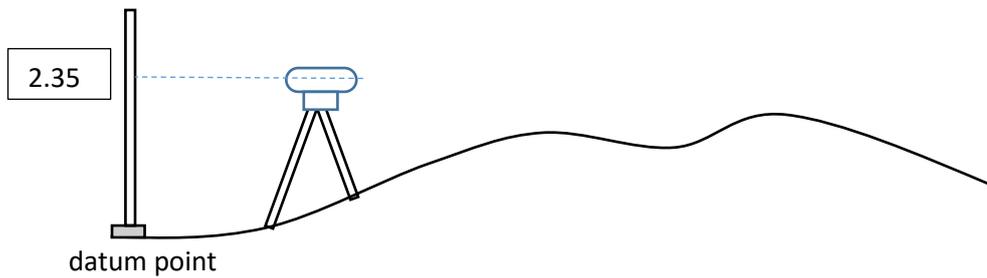
A **level** can be used to measure changes in height. This instrument allows the surveyor to identify a point on a graduated staff at the same elevation as the axis of the viewing telescope.



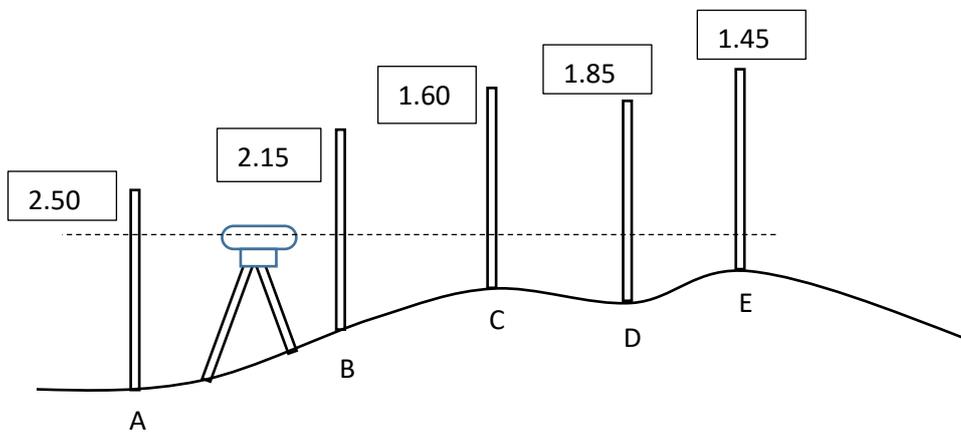
Figure 140: Survey level and graduated staff

The survey begins by selecting a solid reference point which can be used as a datum, to which all height measurements can be related. The datum might, for example, be the top of a brick wall or large boulder.

The level is set up on its tripod. The graduated staff is positioned with its base on the datum point, and a height reading is made.



A series of further points can then be surveyed. In each case, the height reading on the staff is recorded.



At some stage, it may become necessary to move the level to a new location. When this is done, a height measurement taken should be taken for the previously recorded survey point. This will allow a calculation of the relative heights of the sight lines for the different positions of the level. Further points can then be surveyed.

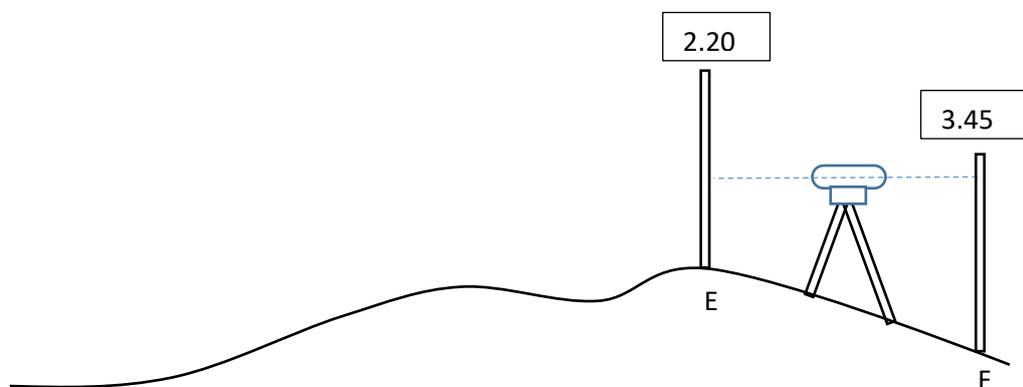


Figure 141: Sequence of measurements during a survey by levelling

It is important to record survey results systematically so that no errors are made in processing the data. A tabular format has been developed for recording levelling data (University of Manchester, 2016), as shown in figure 142. Readings for the example survey are shown:

Back sight	Inter-mediate	Fore sight	Rise	Fall	Reduced level	Distance	Remarks
2.35						6.3	datum
	2.50					8.4	point A
	2.15					7.9	point B
	1.60					9.2	point C
	1.85					8.3	point D
2.20		1.45				6.9	point E
		3.45				10.4	point F

Figure 142: Field data for the levelling survey

At each location of the level, the first reading taken is called the **back sight**. The last reading taken before the level is moved to its next location is called the **fore sight**. Any additional readings taken while the level is in a particular location are recorded as **intermediate readings**.

Each row in the table refers to a particular survey point. The height of point E was measured twice, with the level in different locations along the transect line. Both fore sight and back sight heights are therefore recorded.

The distances between successive survey points along the transect line are measured by tape and recorded in the **distance** column.

The data can then be processed to determine the actual heights of the survey points. We will assume that the datum point is at a height of 100m above sea level. The back sight on the datum point gave a staff reading of 2.35m.

Back sight	Inter-mediate	Fore sight	Rise	Fall	Reduced level	Distance	Remarks
2.35					100.00	6.3	datum
	2.50			0.15	99.85	8.4	point A
	2.15		0.35		100.20	7.9	point B
	1.60		0.55		100.75	9.2	point C
	1.85			0.25	100.50	8.3	point D
2.20		1.45	0.40		100.90	6.9	point E
		3.45		1.25	99.65	10.4	point F

Figure 143: Calculation of the elevations of survey points

The first survey point A gave a height reading of 2.50m. There is a difference in elevation of 0.15m compared to the back sight reading on the datum point. A larger reading on the measuring staff represents a lower ground level, so this figure is recorded in the **fall** column of the table. Applying a fall of 0.15m to the datum height gives a new elevation of 99.85m, which is recorded in the **reduced level** column for point A.

Point B was surveyed next. The staff reading was 2.15m, compared to a reading of 2.50m for the previous point A. This represents a difference in elevation of 0.35m, but this time the ground has risen between survey points. The result is recorded in the **rise** column, and applied to the previous elevation of point A to obtain the new reduced level value of 100.20m.

The calculation proceeds in the same way up to point E, where the location of the level was changed. We start a new sequence of calculations at point F. The staff reading of 3.45m at F is compared with the back sight reading on point E of 2.20m, showing a fall of 1.25m. This difference can be subtracted from the reduced level at point E to obtain the new elevation at point F.

Once all elevations have been calculated, a ground profile can be plotted using the horizontal distances measured between the survey points.

The levelling technique described above is suitable for obtaining the ground profile along a single transect line: for example, across a beach and sand dune system in a coastal geomorphology study. Often, however, an accurate plan or map of a land area also needs to be produced. For this, the technique of **triangulation** is suitable.

The area to be mapped is marked out as a series of large triangles using ranging poles, as in the beach survey illustrated in figure 145. A side of one triangle is taken as a **base line**, and its compass bearing is measured. This will allow the final map to be correctly oriented.

Distances are measured between the survey points by tape, keeping the tape at the same height above the ground at each point. The height of each survey point relative to a datum is found by levelling, using the method described earlier in this section.

Where changes of height occur between survey points, the horizontal distance is calculated by Pythagoras' theorem:

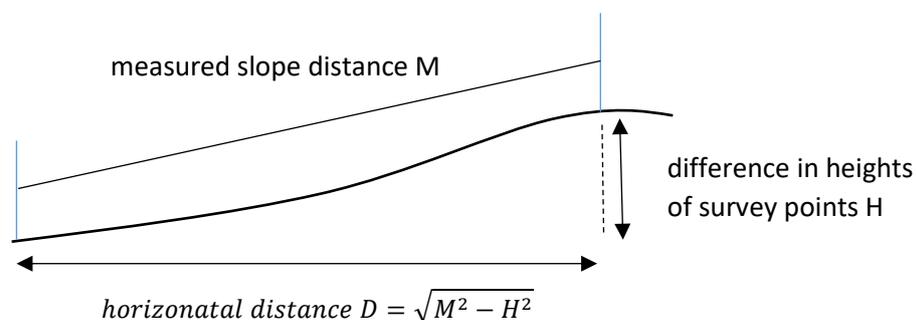


Figure 144: Determining horizontal distances between survey points

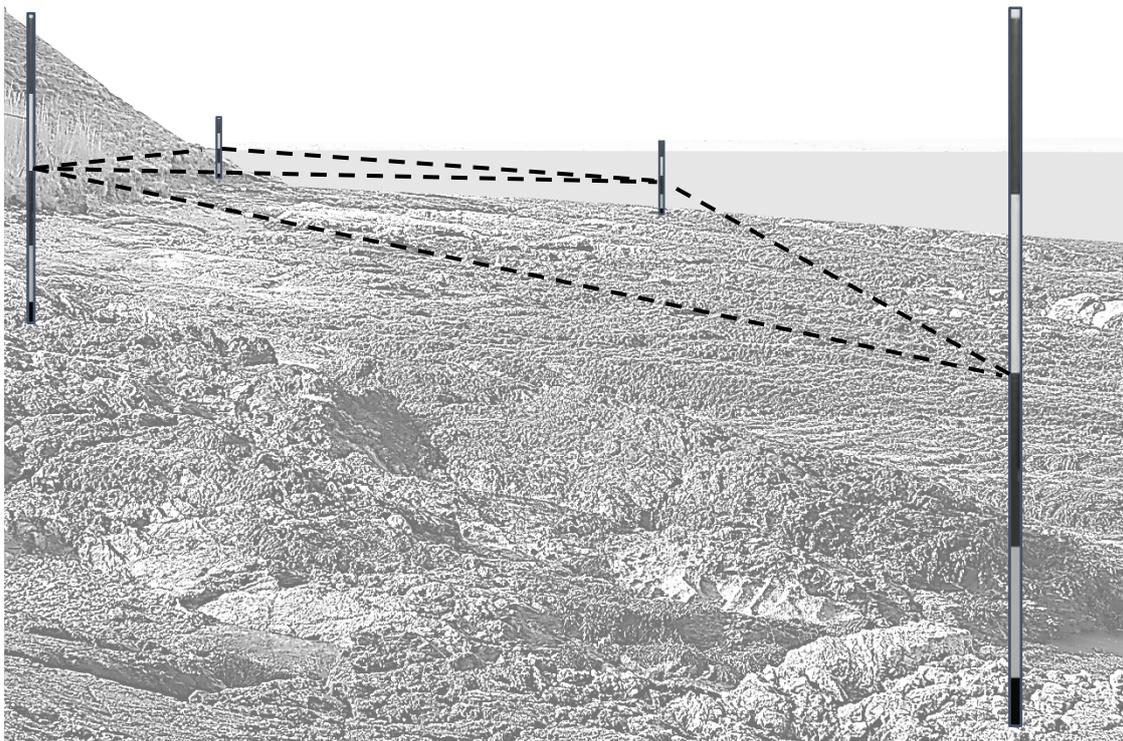
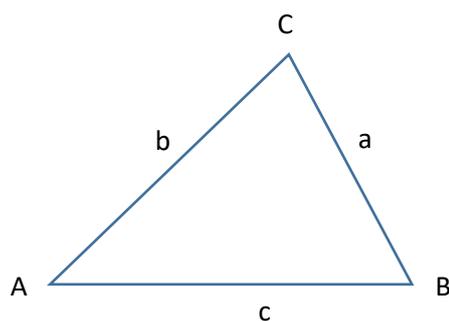


Figure 145: Triangulation method for mapping a beach

The angles of the triangles can now be calculated, to assist in plotting a scale plan or map of the survey area. If the lengths of the three sides of a triangle are known, the angles can be found by applying the cosine rule. Labelling the sides of the triangle as **a**, **b** and **c**, and the opposite angles **A**, **B** and **C**:



then:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

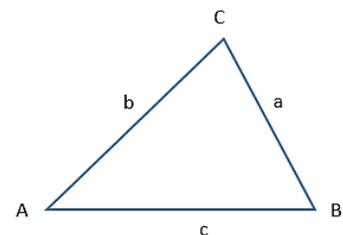
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Figure 146: Use of the cosine rule to find the angles of a triangle with known lengths of sides

It is convenient to set up a spreadsheet so that angles can be calculated directly from the measured slope distances and elevations of the survey points.

	A	B	C	D	E	F	G	H	I	J
1										
2		triangle corner	survey point name	elevation (m)	angle (degrees)					
3	A		point 9	23.67	100.3					
4	B		point 11	31.49	50.0					
5	C		point 12	28.23	29.7					
6				angle total:	180.0					
7										
8		triangle side	distance (m)	change in height (m)	horizontal distance (m)					
9	a		19.56	-3.26	19.29					
10	b		15.22	4.56	14.52					
11	c		9.86	7.82	6.01					
12										



Formulae:

	A	B	C	D	E
1					
2		triangle corner	survey point name	elevation (m)	angle (degrees)
3	A		point 9	23.67	=ACOS(((C10*C10)+(C11*C11)-(C9*C9))/(2*C10*C11))*180/PI()
4	B		point 11	31.49	=ACOS(((C9*C9)+(C11*C11)-(C10*C10))/(2*C9*C11))*180/PI()
5	C		point 12	28.23	=ACOS(((C9*C9)+(C10*C10)-(C11*C11))/(2*C9*C10))*180/PI()
6				angle total:	=SUM(E3:E5)
7					
8		triangle side	distance (m)	change in height (m)	horizontal distance (m)
9	a		19.56	=D5-D4	=SQRT(C9*C9-D9*D9)
10	b		15.22	=D5-D3	=SQRT(C10*C10-D10*D10)
11	c		9.86	=D4-D3	=SQRT(C11*C11-D11*D11)
12					

Figure 147: Spreadsheet for processing triangulation data

As an example of the use of surveying, we describe two projects carried out by members of an industrial archaeology course. In the first, a survey was carried out to record details of an old winding drum on a gravity worked incline at the Dinorwig slate quarry, Llanberis. The drum house chosen for the project, at the head of the C5 incline, is now in a derelict condition, but much of the machinery remains in place and can be measured and photographed.

Old photographs (Carrington, 1994) show the drum house and incline, towards the end of the period when it was in use (figure 148). The building contains a large wooden drum on which steel cable was wound. Trucks of slates could be attached to one end of the cable, then lowered down the incline by releasing the brake on the drum. At the same time, the other end of the cable would be wound back onto the drum, hauling empty trucks up the incline in the process.

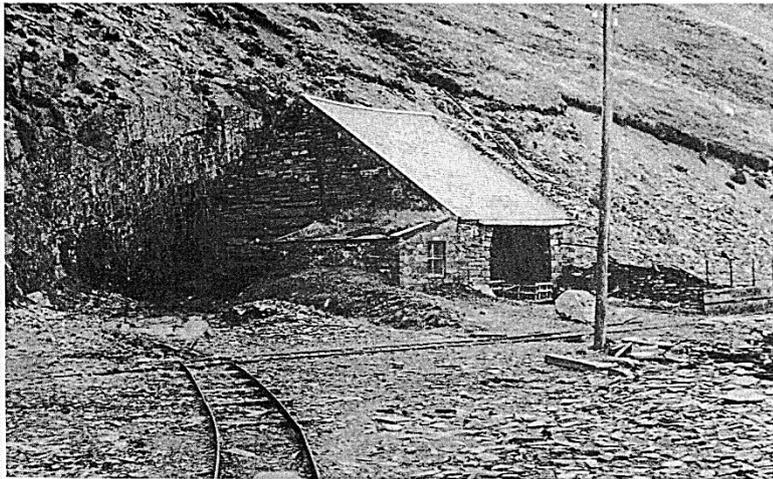
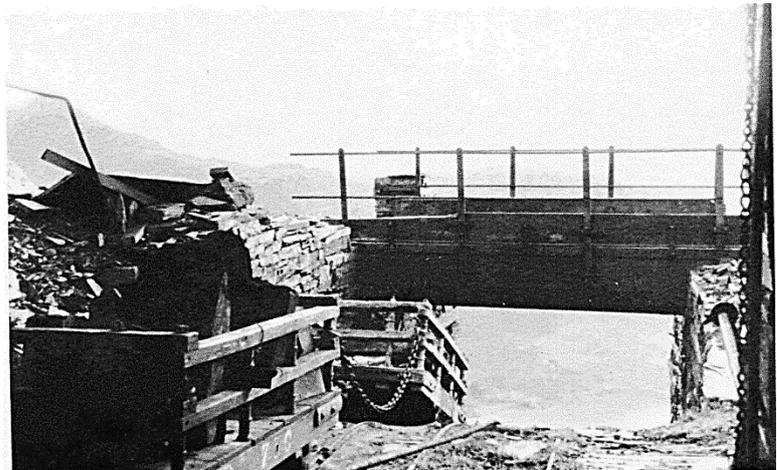
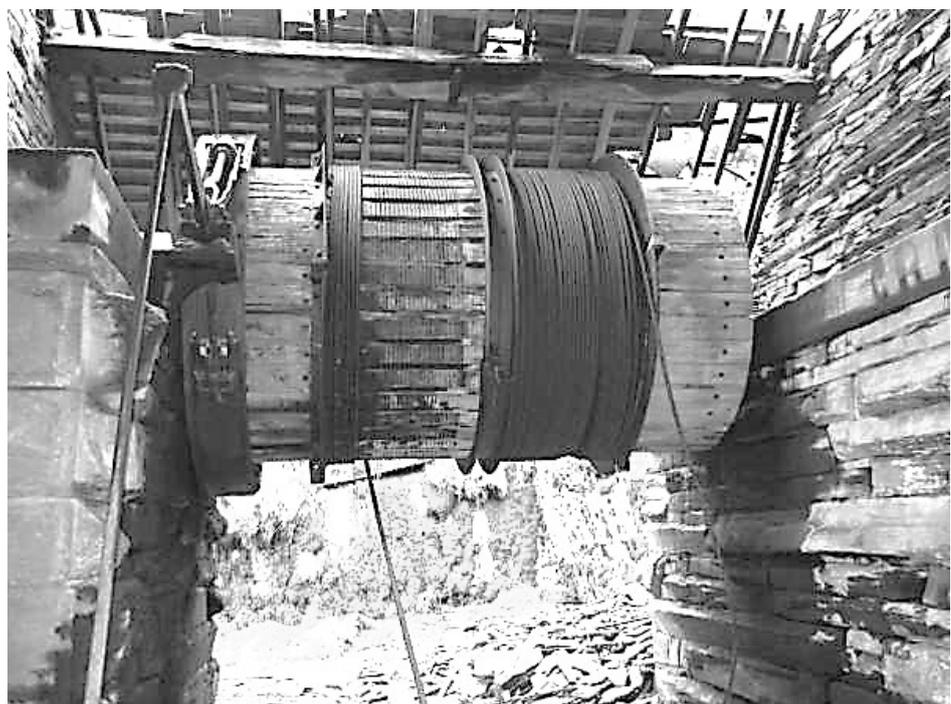


Figure 148: Winding drum,
Dinorwig quarry



An initial survey of the site was carried out by levelling and triangulation, so that plans and elevations could be drawn (figure 150). Detail of the machinery was measured in situ, with additional dimensions obtained from a series of photographs taken from different positions, as in figure 149. Where the designs of missing elements of the building or machinery could be determined, these were added to the drawings.

Figure 149:
Winding drum and
brake mechanism



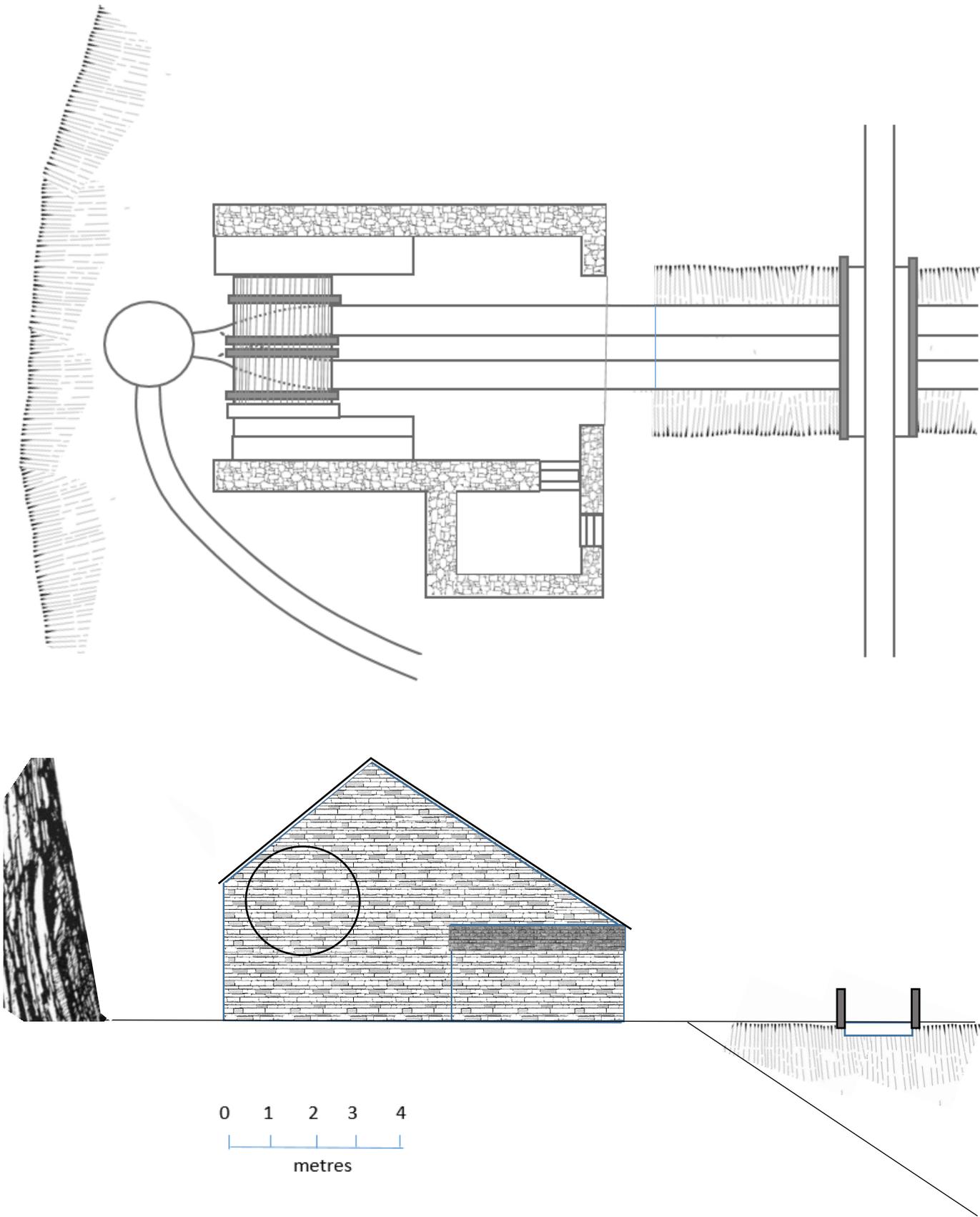


Figure 150: Plan and elevation of the C5 winding drum, Dinorwig quarry

The second industrial archaeology project involved the mapping of tunnels and a unique underground waterwheel at the Ystrad Einion lead mine near Machynlleth in mid-Wales. This small mine opened 1877 and operated for only about 20 years (Bick, 1976), but was equipped with some of the most modern mining technology of the time.

The group carried out surveying of the adit tunnel and chambers on the No. 3 level. A plan was produced by a combination of triangulation and the measurement of compass bearings (figure 153).



Figure 151: Adit tunnel leading to the underground waterwheel

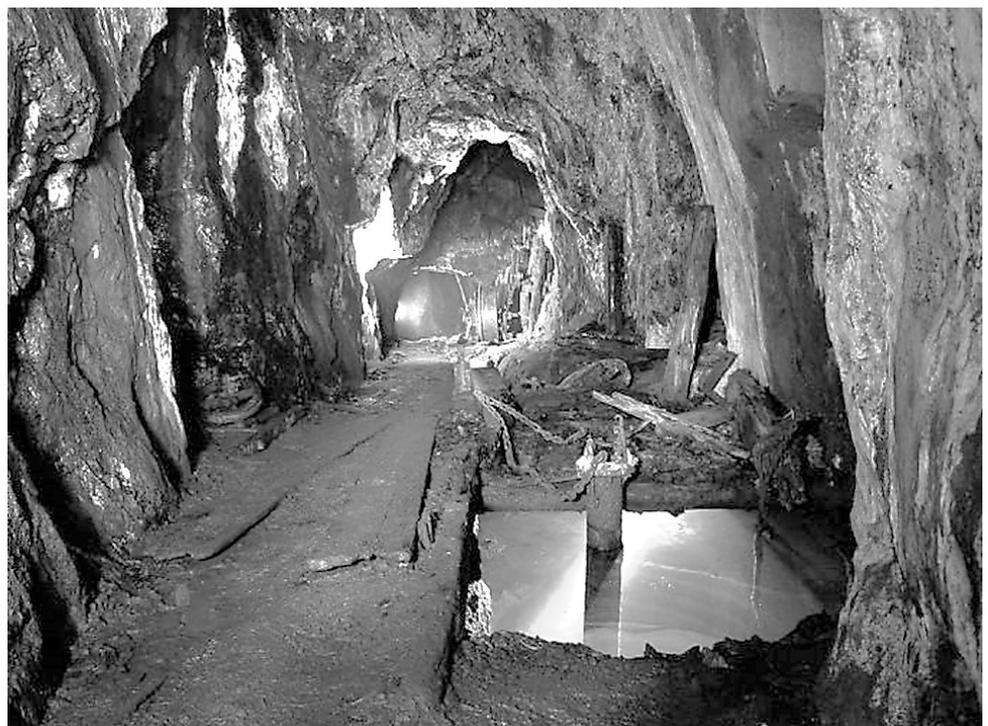


Figure 152: Flooded shaft with pump tube and remains of headframe for hauling up ore from the lower workings.

The water wheel is in the chamber in the distance.

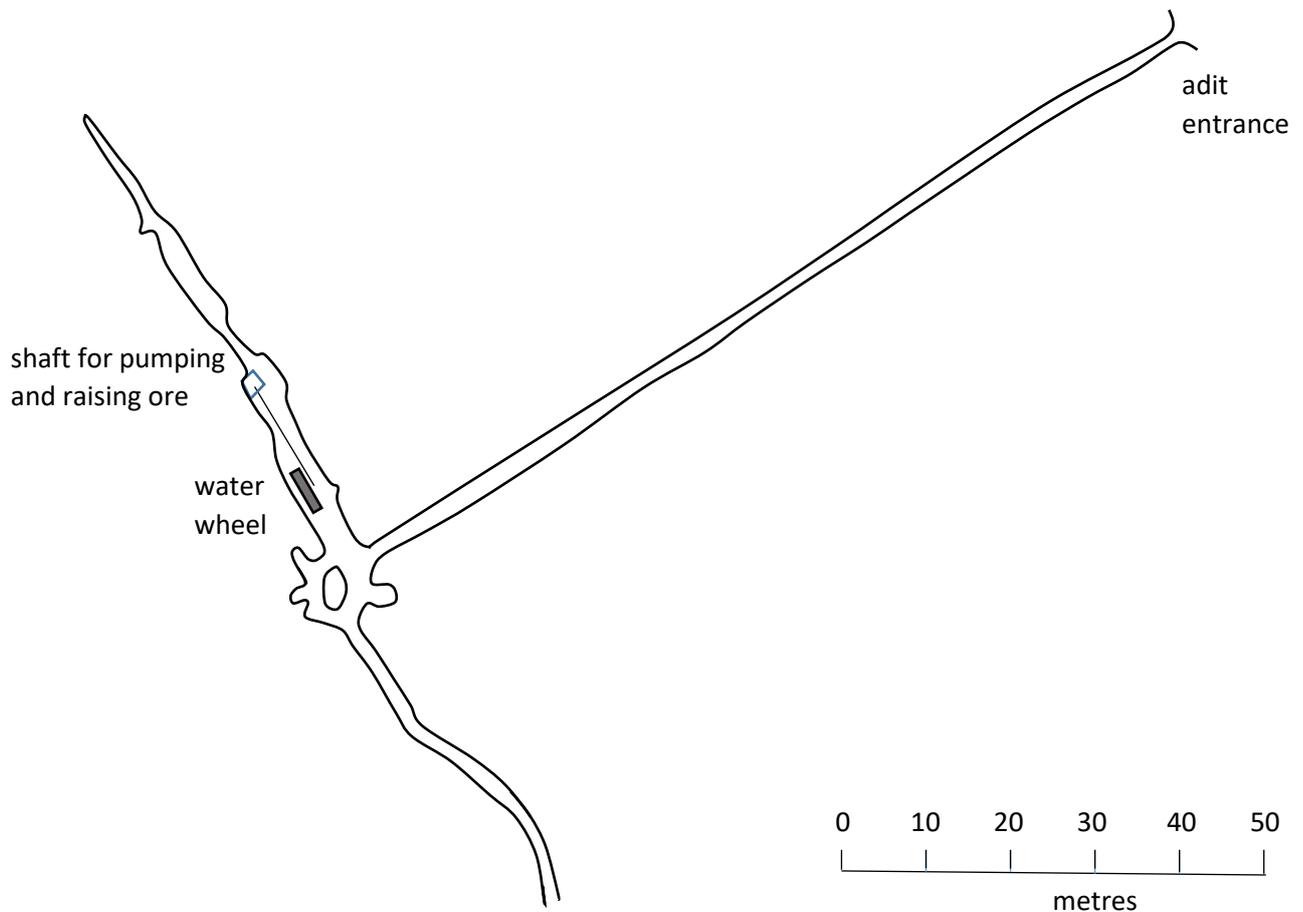


Figure 153: Plan of workings on the No. 3 level, Ystrad Einion mine

The main feature of interest in the mine is the well preserved underground water wheel, 16 feet in diameter (figure 154). The wheel provided motive power for the twin functions of pumping water from the lower levels, and raising buckets of lead ore. Water was carried to the wheel through a wooden trough from a higher level of the mine. Drainage water and water discharged from the wheel flowed out of the mine through the adit tunnel.

The water wheel was measured in situ using a tape and a graduated surveying staff, making sketches to indicate the positions of the recorded dimensions. A number of photographs were taken of the machinery from different directions. Some components had become disassembled or were missing, but members of the group were able to use their knowledge of mechanical engineering to deduce how the mechanisms operated.

The water wheel turned in an anti-clockwise direction when viewed from the front. The wheel itself is constructed from cast iron segments connected to the central axle by timber spokes. The segment plates hold the timber buckets of the wheel. The whole structure is supported on a sturdy wooden trestle.



Figure 154: Underground water wheel, Ystrad Einion mine.

The winding drum for raising ore is seen on the left, and was powered by the large gear wheel.

A crank attached to the water wheel was linked by chain to the pump rods in the mine shaft.

The wheel carried out its pumping function by means of a rotating crank attached to the main axle. The crank provided a reciprocating motion for a chain, which passed over pulleys to an angle bob above the mine shaft. This converted the motion into a vertical lifting force to operate pump rods in the shaft.

The water wheel has a winding drum for hauling buckets of ore up the shaft. The drum was powered by reduction gearing from the water wheel. A mechanism allowed the drive to the drum to be started and stopped by moving the drum shaft sideways to engage or disengage metal lugs on a clutch plate. A metal band brake was fitted around the edge of the winding drum.

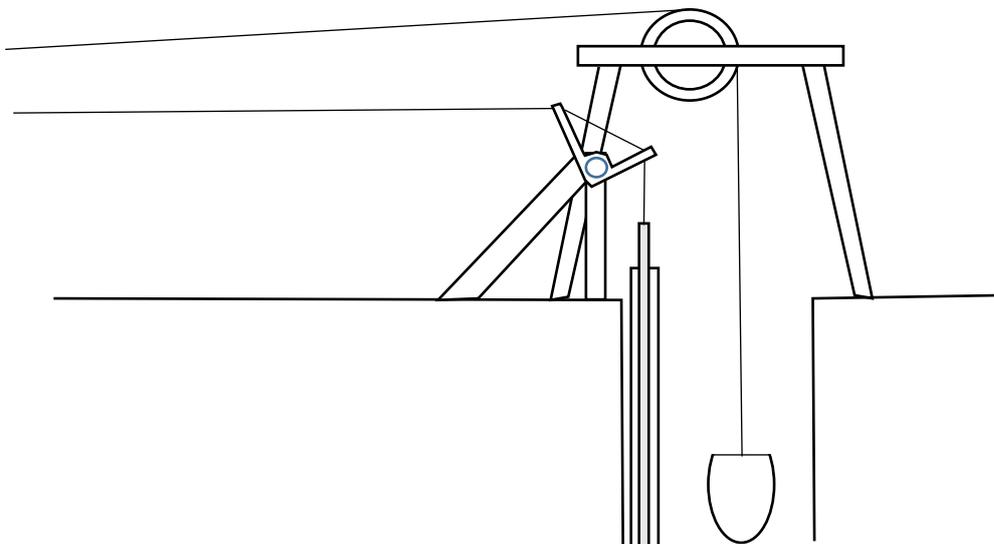
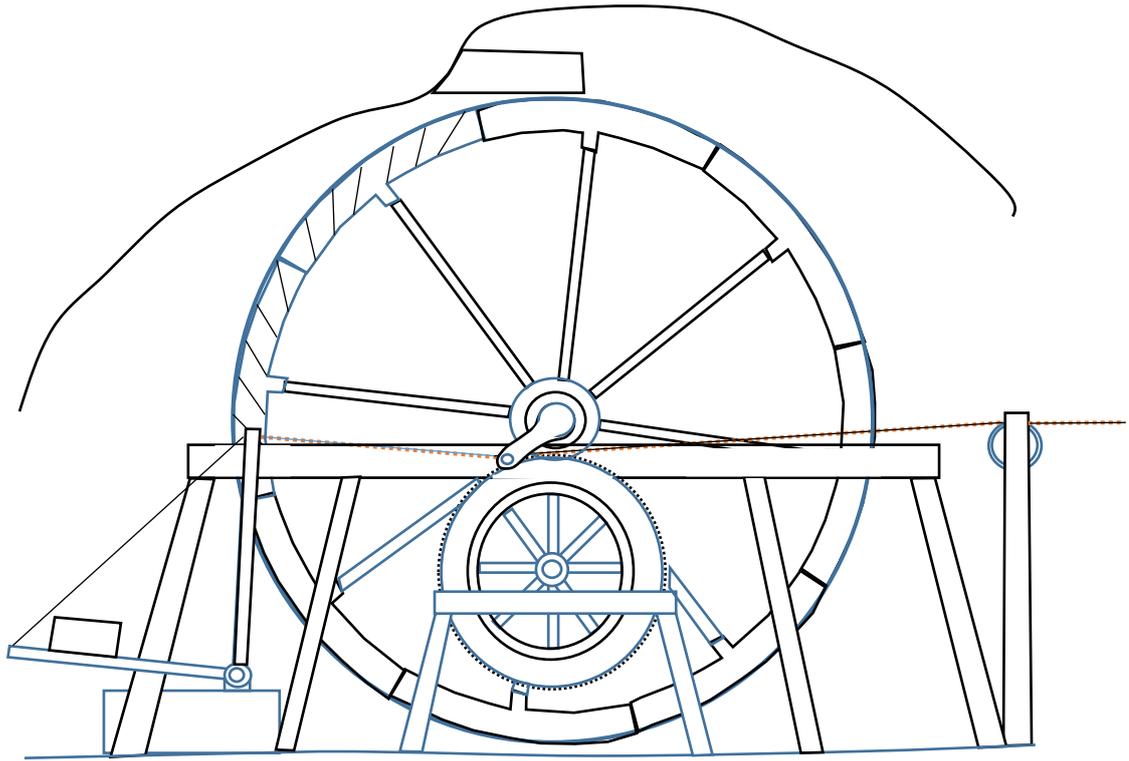


Figure 155: Arrangement of the pumping and haulage shaft



1 metre

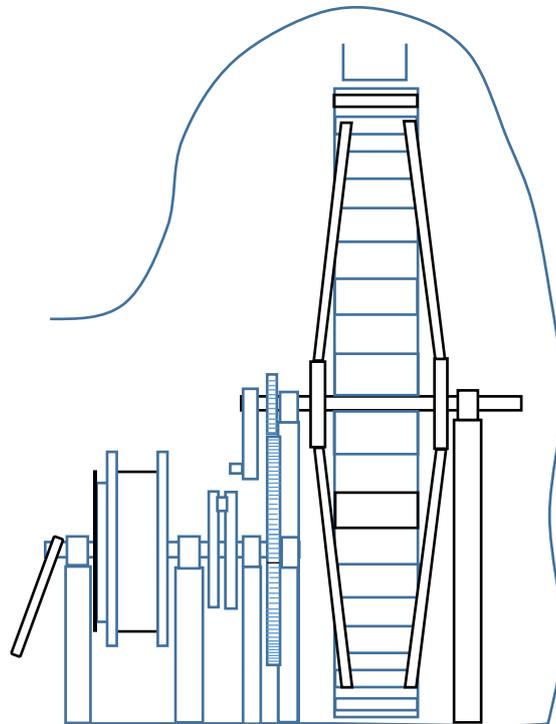


Figure 156: Underground water wheel at Ystrad Einion mine

Engine rotation

An important application of numeracy is in the production of computer aided designs for machinery, often with the addition of animation to illustrate the motion of the machine components. In the final sections of this chapter, we will examine some of the mathematical techniques for representing objects in computer programs and for calculating movement.

In two dimensions, a point can be located by its horizontal x-coordinate and its vertical y-coordinate. Geometrical shapes can be specified by a combination of co-ordinates, for example: the centre position and radius for a circle, or the corner positions of a triangle or rectangle. As an example of a two dimensional graphics application, we will consider how an animation of a diesel engine could be produced in a computer program.

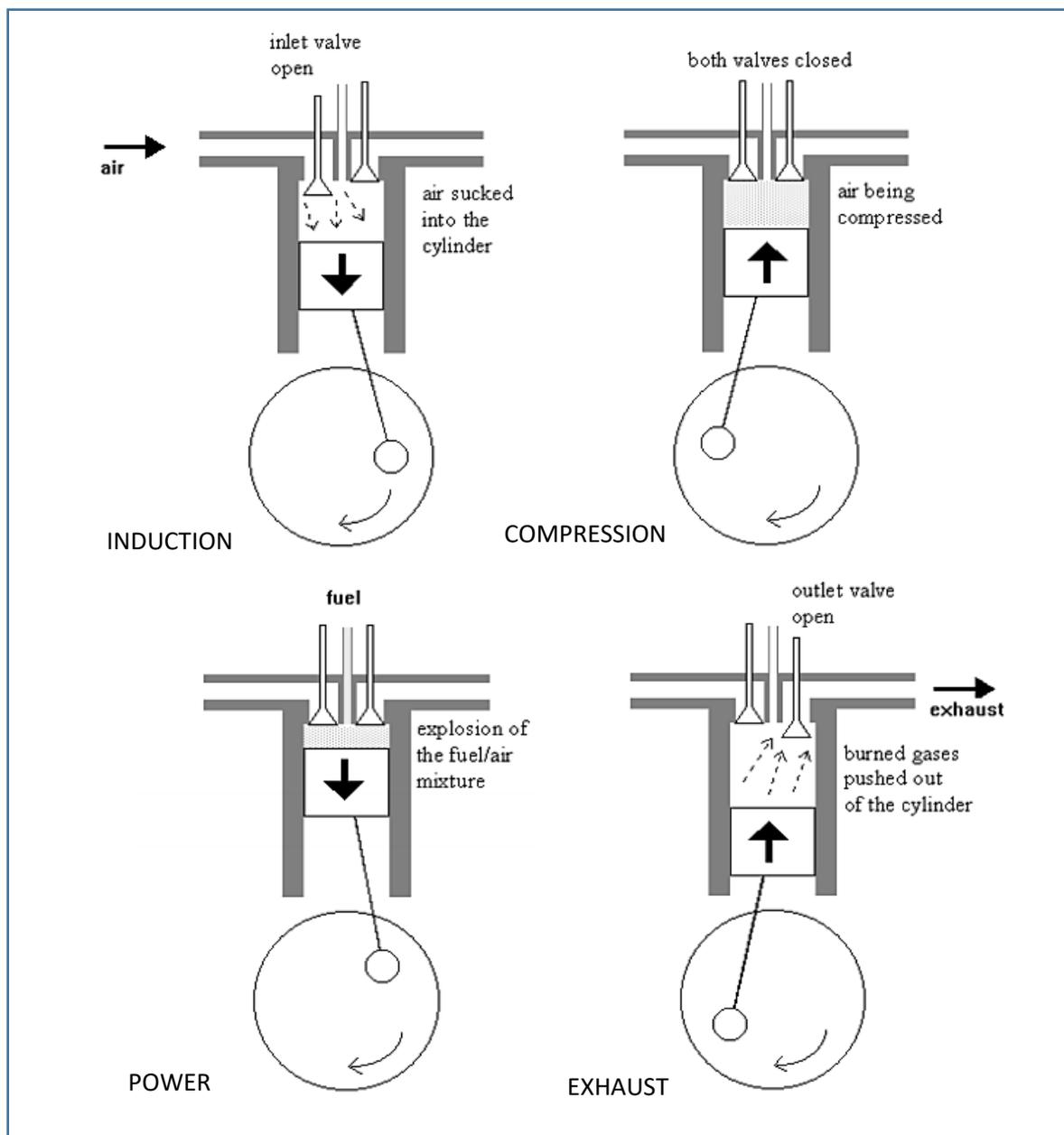


Figure 157: Operating cycle for a diesel engine

The operating cycle for a diesel engine is illustrated in figure 157. The piston in the cylinder is linked to the rotating crank shaft by the connecting rod. An input valve allows air to enter the cylinder at the start of the cycle. The air is compressed, fuel is injected and the mixture ignites. Expanding gasses drive the piston downwards during the power stroke, then the exhaust valve opens to allow the burned gases to escape. The cycle is then repeated.

To develop a two-dimensional animation of the engine, the fixed dimensions of the linkage must first be specified. The movement of the piston will depend on the radius of rotation of the connecting rod bearing B, and the length of the connecting rod AB.

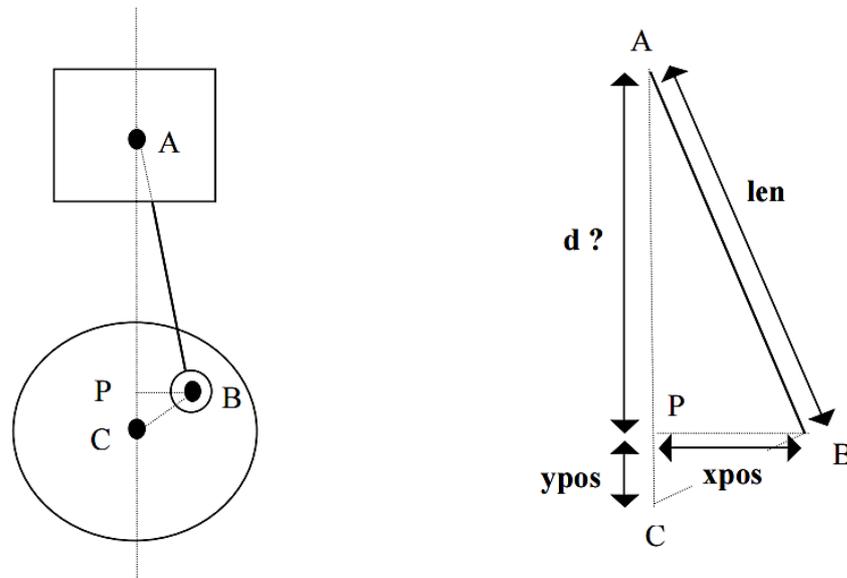


Figure 158: Geometry of the diesel engine motion

An animation can be produced by displaying the piston and connecting rod for a fraction of a second at each successive angle as the crankshaft rotates. The piston and connecting rod will then be redrawn in the next position.

The rotation angle will be measured in degrees clockwise from the positive x-axis. The coordinates of moving parts can then be calculated using geometry and trigonometry.

The vertical centre line of the engine cx and the horizontal centre line of the flywheel cy form the origin for the coordinate system, as shown in figure 158.

Taking radius of crank rotation as r , and the length of the connecting rod as l , then for any rotation angle θ we can calculate $xpos$ and $ypos$ for the crank bearing:

$$xpos = cx + r \cos \theta$$

$$ypos = cy + r \sin \theta$$

The distance of the centre of the piston above the axis of the crank shaft, shown as CA in figure 158, can be calculated:

$$piston\ height = ypos + \sqrt{l^2 - xpos^2}$$

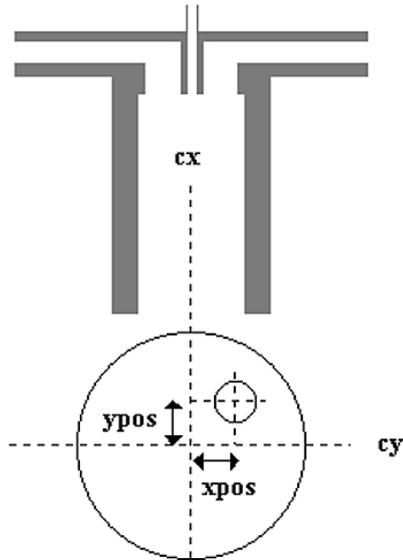


Figure 159:
Co-ordinate system for
the diesel engine model

The final requirement for the simulation is to show the input and exhaust valves opening and closing at the correct points in the engine cycle. The full engine cycle involves two rotations of the crank shaft, making a total angle of 720° . The inlet valve is open for rotation angles between 0° and 90° , reopening at the end of the cycle at angles between 660° and 720° . The exhaust outlet valve is open at angles between 450° and 600° .

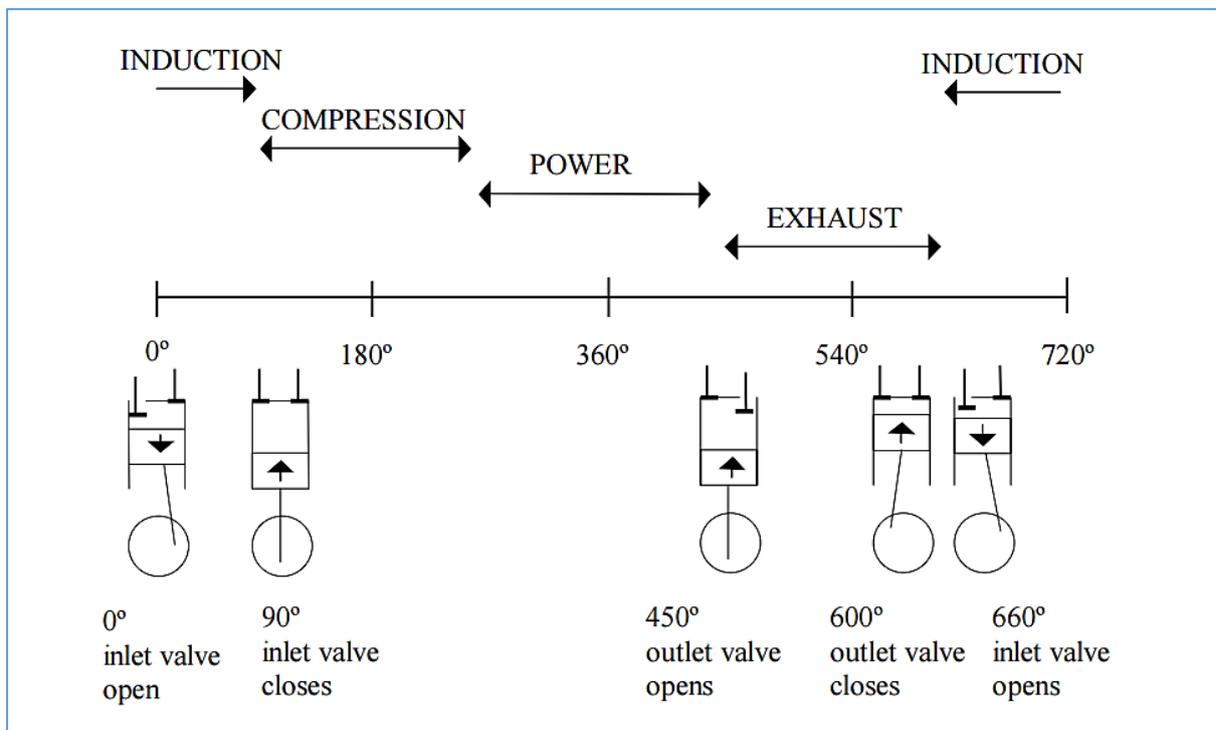


Figure 160: Valve positions during the diesel engine cycle

A computer program can then be written to generate the animation, as shown in the flowchart below.

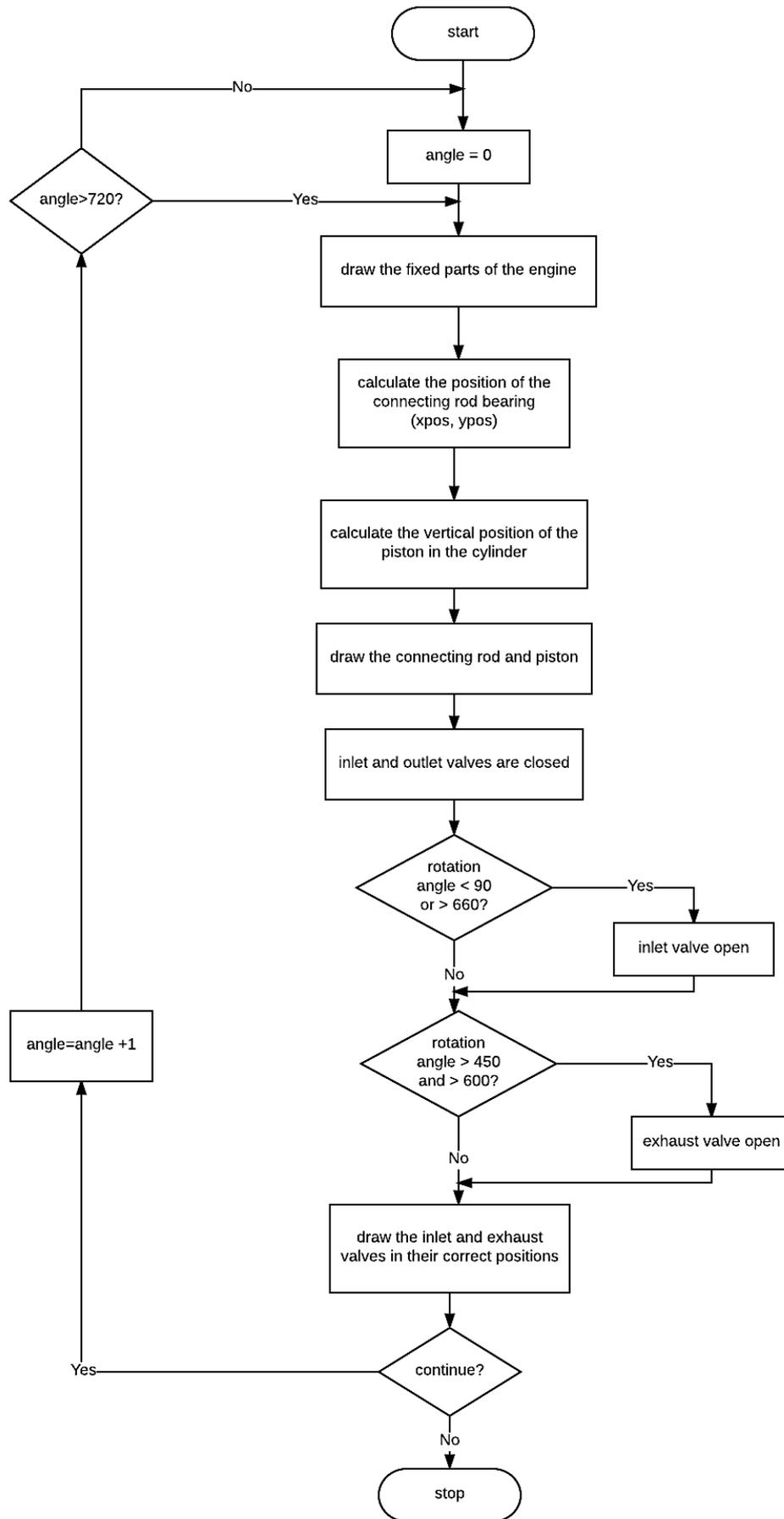
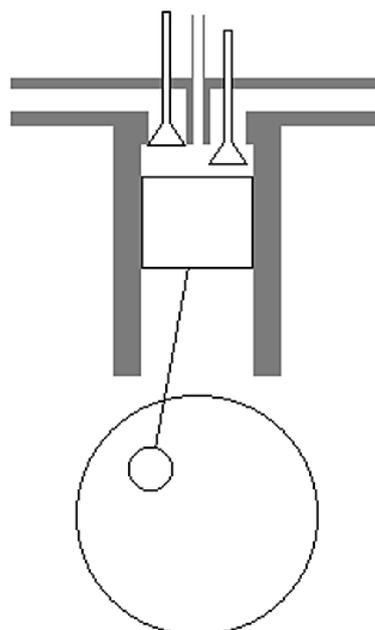


Figure 161: Flowchart for the diesel engine animation

Figure 162:
Representation of the diesel engine
during the exhaust stroke, with the
outlet valve open



Deltic railway locomotive diesel engine

As a convenient alternative to writing a computer program, a computer aided design application with animation facilities can be used to illustrate the motion of machinery. In the next section, we present a project by an engineering student which uses Solidworks CAD software to produce a three-dimensional model of a Deltic diesel engine.

The Deltic engine design is unique in having groups of three double-acting cylinders arranged in a triangular pattern.

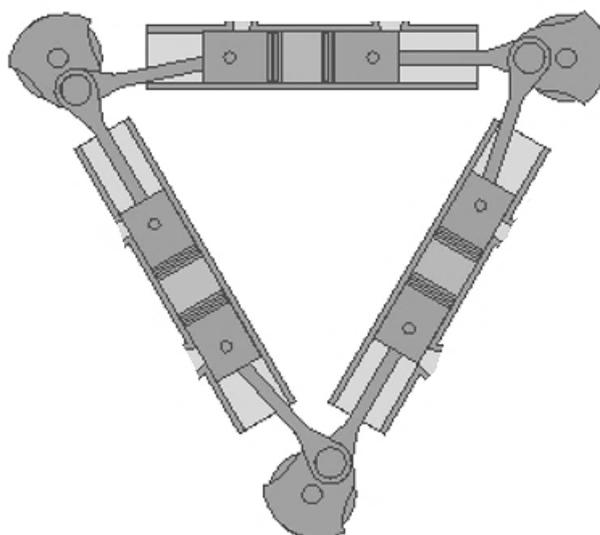


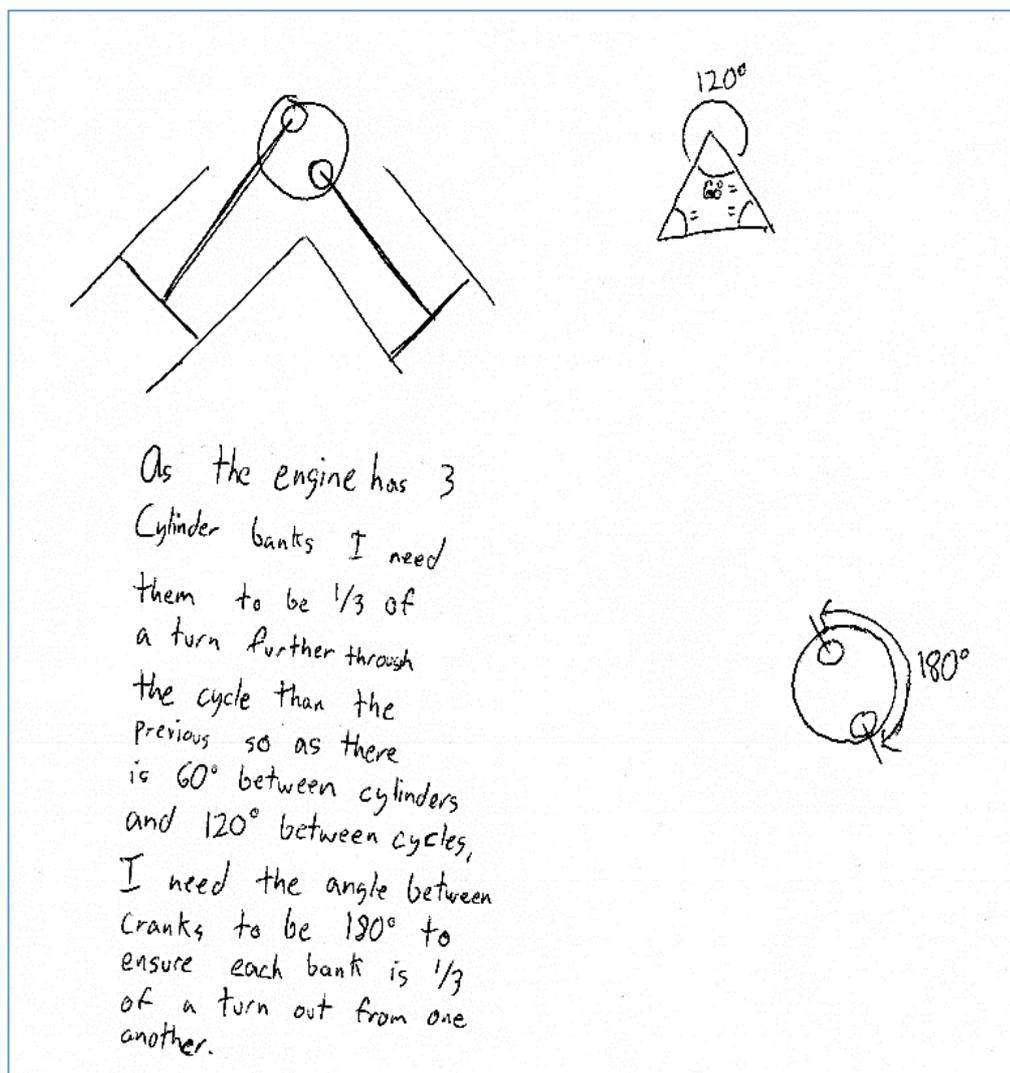
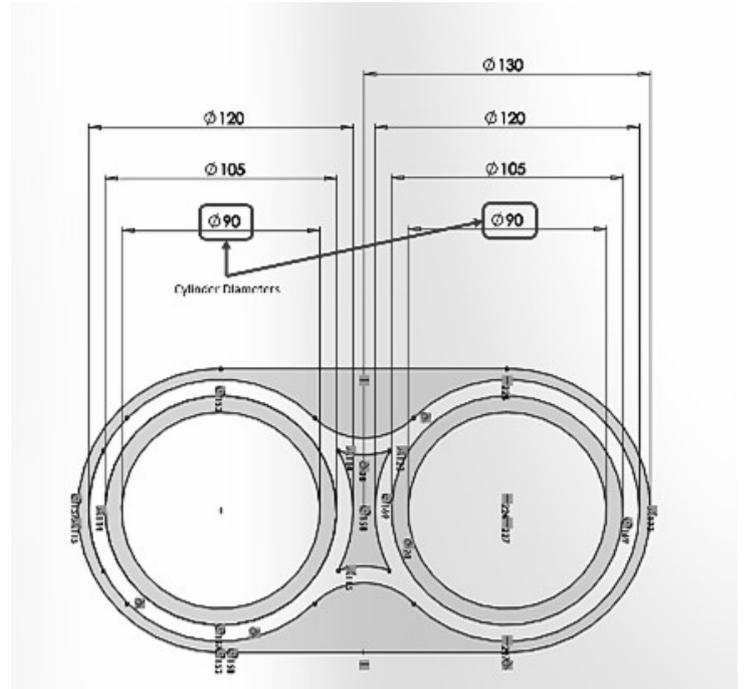
Figure 163:
Triangular arrangement of the
double-acting cylinders of the
Deltic diesel engine

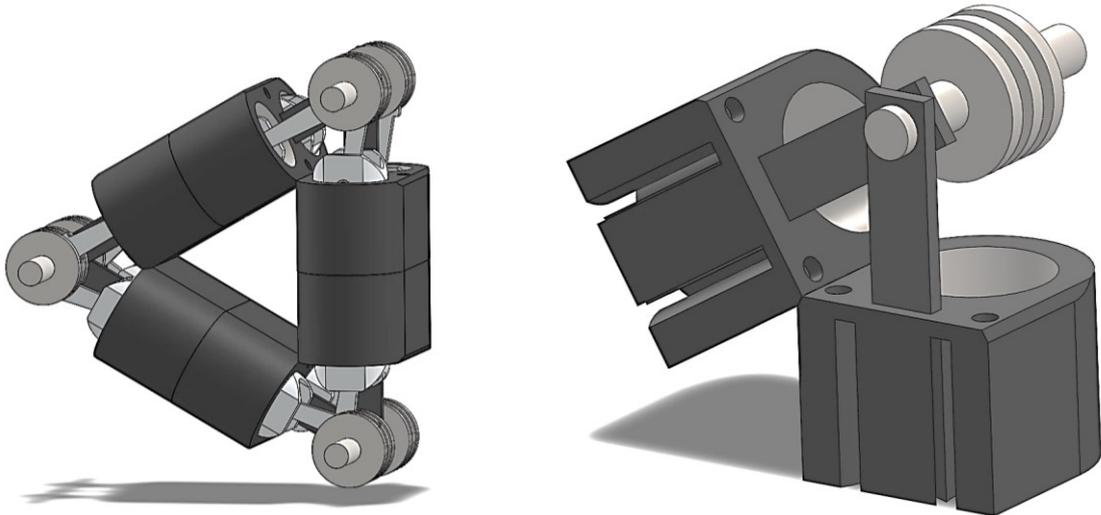
During the power stroke, the two pistons within a cylinder are driven apart, causing rotation of the two crank shafts to which they are connected. This in turn causes compression, induction or exhaust strokes in the other two cylinders. A power stroke occurs in each of the three cylinders in sequence.

The following is an extract from the student's account of his design work:

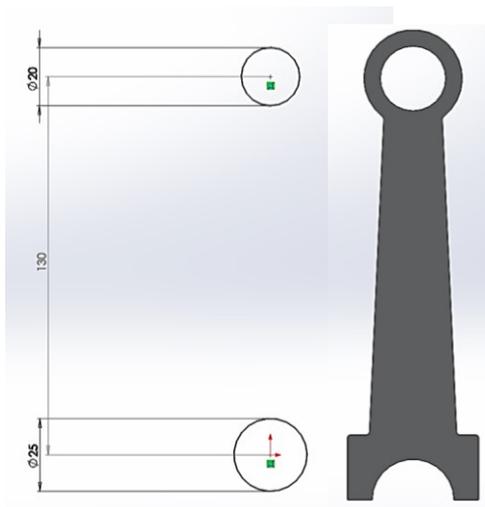
The first thing I designed was the cylinder. Each cylinder bank is split into two separate cylinder blocks mated with a gasket in between them. This is mainly a maintenance friendly design to make it easier and cheaper to replace cylinders.

I decided a diameter of 90mm would be sufficient for the model as it would be large enough to work theoretically if the engine was built. There is a water jacket around the cylinders for cooling purposes.





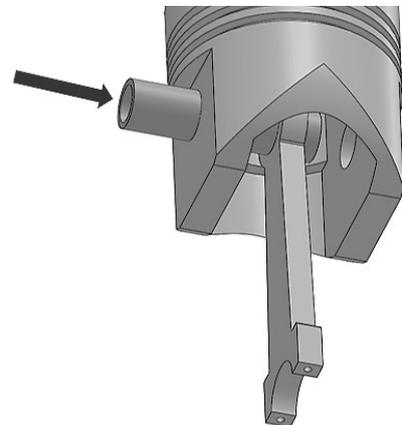
Once I had the crankshafts designed I placed them into the assembly using centre mounts to give the spacing needed for clearances of the connecting rods. When the crankshafts were mounted, I placed the piston at the desired position. I then measured from the piston to the crank to get the length for the connecting rod.



When the length was determined, I started work on designing the connecting rod. I made two circles at the correct distance and the right sizes for the crank and gudgeon pin (holds the piston to the connecting rod). I then drew the rest of the connecting rod but I cut the crank side in half in order to place it on the crankshaft. The piston side doesn't need to be split as the gudgeon pin is pressed through the piston and connecting rod.

Now that the main moving components were designed, I could assemble it into a functional model in the CAD software. I first placed all the moving components into the model and mated them to the surfaces that they needed to be in contact with.

Once all the components were in place I put the engine in time and prevented any movement until I'd made a gearing system to transfer the power to a single output. I made two support plates to hold a PTO shaft in the centre and support the crank shafts on both sides.



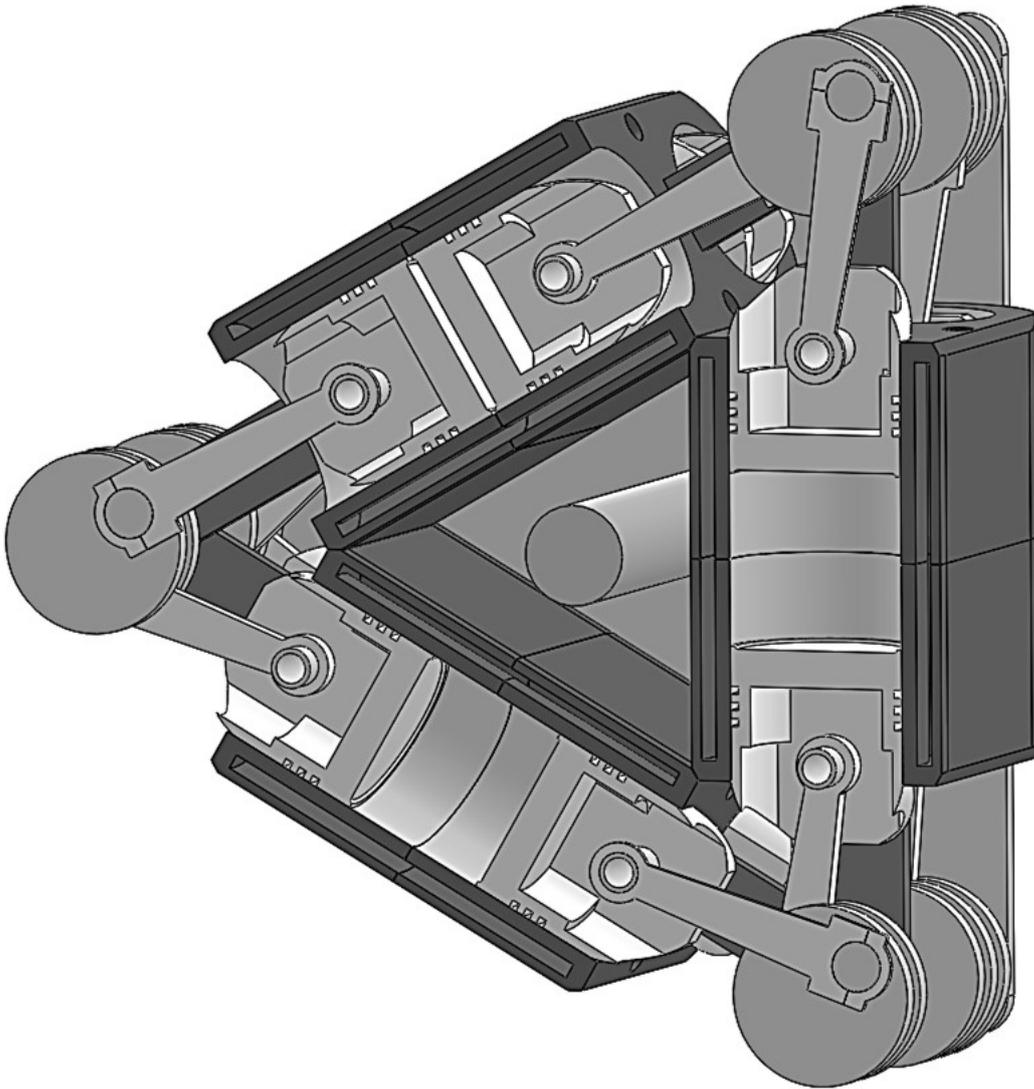


Figure 164: The completed engine animation

This project provides a good example of the integration of numeracy into vocational course activities. A range of skills have been demonstrated within the wider definition of numeracy:

- Problem solving, in deducing the correct sequence of motion of the engine components
- Mathematical techniques, in calculating appropriate dimensions and movement paths for components
- Application of techno-mathematical literacy in the use of computer aided design software.
- Use of relevant vocational knowledge. The student was able to apply his understanding of the operation of a simple diesel engine to develop the designs for a more complex system.
- Communication. Using effective animation techniques to help explain the design to others.

Isometric projection

The technique described earlier for producing computer animations in two dimensions can be extended in a straightforward way to three dimensions by using a coordinate system with three perpendicular axes:

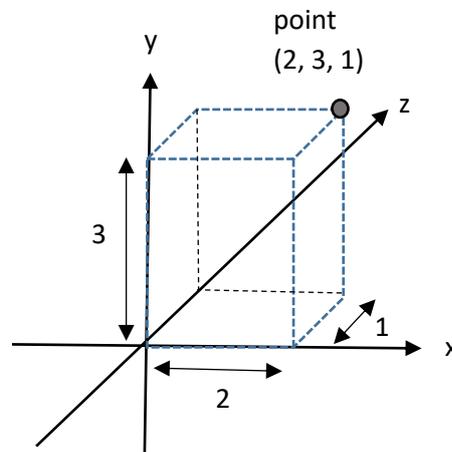


Figure 165: Representing a point in three dimensions

Each **point** is represented by a set of three coordinates, as in this example of a point at the position $x=2$, $y=3$, $z=1$.

Points can be combined to produce **faces** by specifying the series of points making up the perimeter.

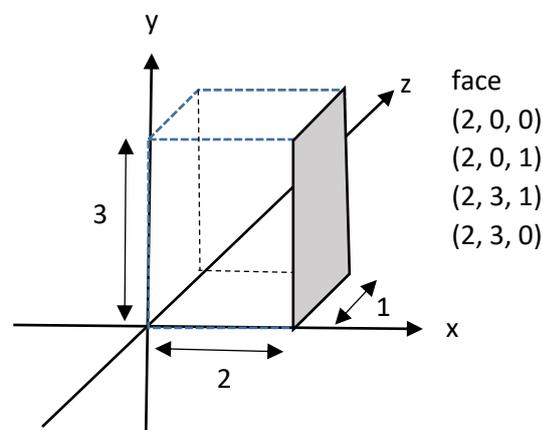


Figure 166: Representing a face in three dimensions

Once a series of faces have been defined for an object, these then have to be represented on a computer screen.

We are familiar with the perspective views of objects seen in photographs, where parallel lines such as the edges of a road or building appear to converge as they extend into the distance. Perspective can be represented in computer generated images. However, this adds extra complexity to graphics calculations. Many computer aided design systems

instead use **isometric projection**, in which parallel edges of objects remain parallel with distance. This generally provides a satisfactory representation of machines which would be observed from a close view point.

Isometric projections can be drawn by hand using special graph paper. The three coordinate directions are represented by sets of parallel lines: the x- and z- directions are inclined at 30° to the horizontal, whilst the y-direction is vertical. A point is selected as the origin of the coordinate system, then the grid used to locate positions of points and faces.

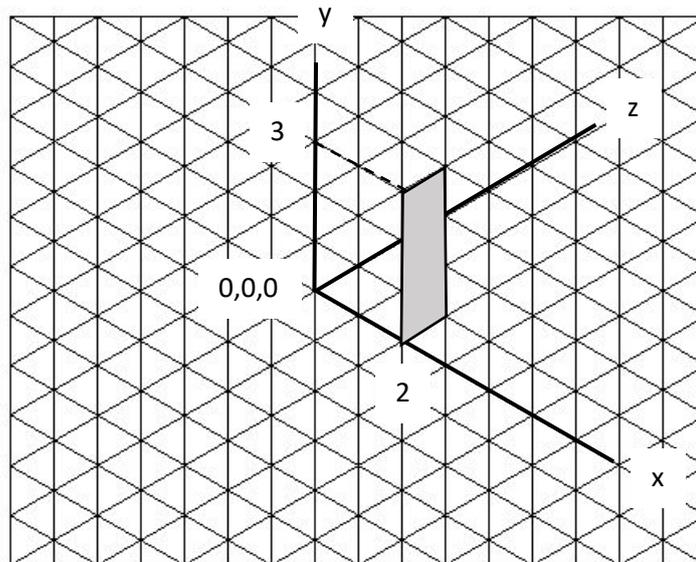
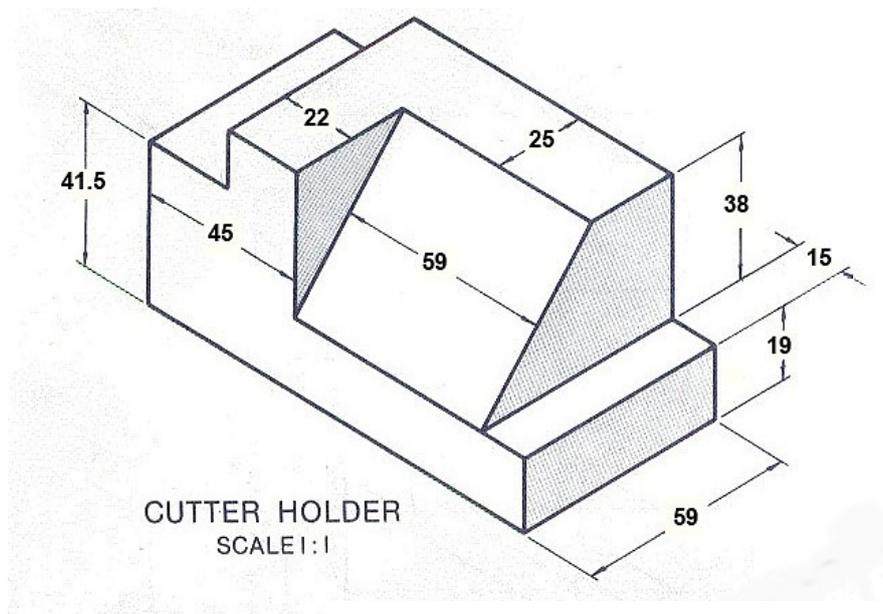


Figure 167: Representation of a face in isometric projection

Complex shapes can be produced in isometric projection, as in this example which includes an inclined face



jmcintyre.wikispaces.com/TDJ3M_Views_and_S

Figure 168: Example of a machine component in isometric projection

A problem which arises in computer applications is that faces must be added in the correct sequence if they are to overlap correctly in the final image. This is achieved by a **depth sort** algorithm. The faces furthest from the viewer need to be drawn first, then the nearer faces plotted on top to build up the picture. Taking an example of a cube:

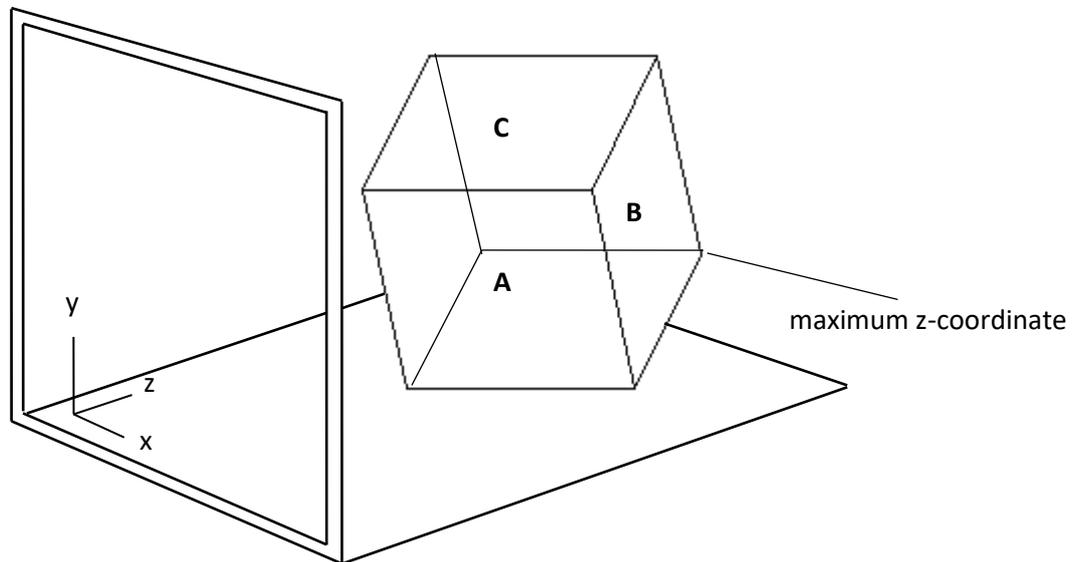


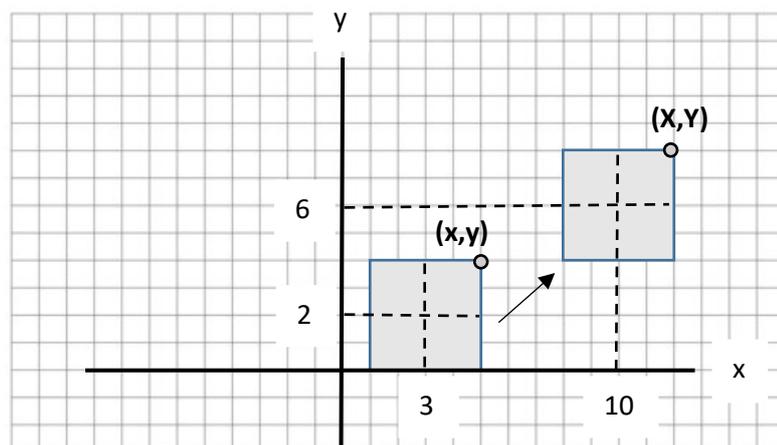
Figure 169: Depth sorting of the faces of a cube

The order in which faces should be plotted depends on the z-coordinates. In this example, **face B** should be plotted first as it is furthest from the viewer and contains the **maximum z-coordinate** of any corner of the cube. **Faces C and A** have **lower z-coordinate values** for the corners, so will be plotted on top. Face B will actually be hidden from view in the finished picture of the cube.

Geometric transformations

In an animation of a machine, we may need to move or rotate the image. Simple movement, known as **translation**, can be achieved by adjusting the coordinate values. For example, suppose that we wish to translate a shape in the x-y plane so that its centre moves from the point (3, 2) to the point (10, 6)

Figure 170:
Translation of a shape
in the x-y plane

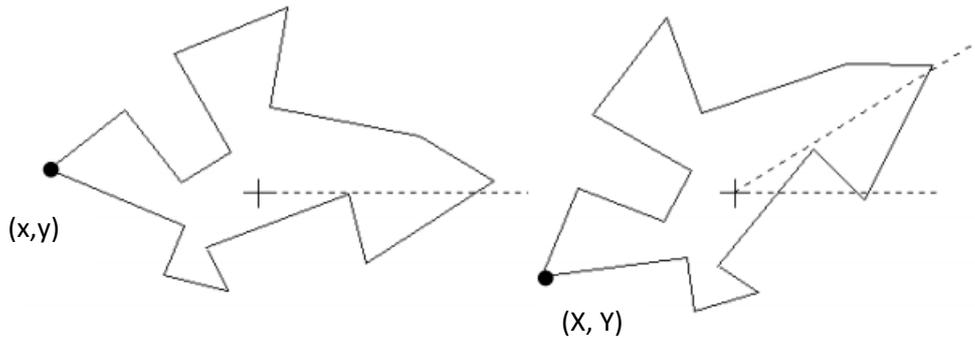


Any general point (x,y) will be moved to the new point (X,Y) according to the equations:

$$X = x + 7$$

$$Y = y + 4$$

Rotation can be handled in a similar way. Suppose that we wish to rotate a shape around the origin of the x - y plane by an angle θ :



As a result of the rotation, a typical point (x,y) on the shape moves to the position (X,Y) . Using trigonometry, it is possible to calculate the new coordinates X and Y :

$$X = x \cos \theta - y \sin \theta$$

$$Y = x \sin \theta + y \cos \theta$$

These equations are derived by means of the trigonometric identities for compound angles:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

which may be found in A-level mathematics textbooks.

Suppose that a rotation by an angle θ transforms a general point from (x,y) to (X,Y) :

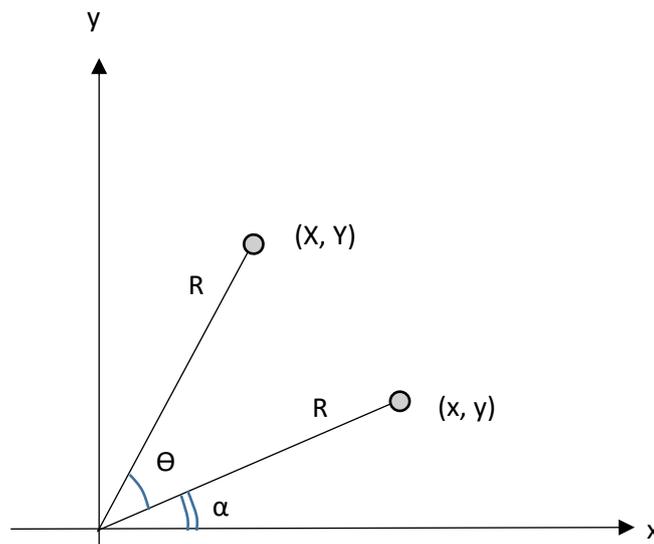


Figure 171: Rotation of a point in the x - y plane

The distance R of the point from the origin will remain the same during the rotation. Therefore, if the initial angle to the point is α

$$x = r \cos \alpha$$

$$y = r \sin \alpha$$

After rotation, the coordinates of the point have become:

$$X = r \cos(\alpha + \theta)$$

$$Y = r \sin(\alpha + \theta)$$

Using the compound angle identities:

$$X = r \cos \alpha \cdot \cos \theta - r \sin \alpha \cdot \sin \theta$$

$$Y = r \sin \alpha \cdot \cos \theta + r \cos \alpha \cdot \sin \theta$$

Substituting $x = r \cos \alpha$, $y = r \sin \alpha$ gives:

$$X = x \cos \theta - y \sin \theta$$

$$Y = x \sin \theta + y \cos \theta$$

as shown on the previous page. These equations can also be written in matrix form:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The equations and the matrices are just different ways of saying the same thing. From the rules for multiplying matrices, it is always the case that:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x.A + y.B \\ x.C + y.D \end{bmatrix}$$

We now have the necessary equations to rotate objects around the origin. However, the situation is a little more complex if it is necessary to rotate an object about a point which is not at the origin.

To achieve a rotation by an angle θ about a point (A,B) , the strategy needed is:

- Find the coordinates of all points on the boundary of the shape relative to the required rotation centre (A,B) .
- Translate the points to the origin $(0,0)$ by subtracting x -coordinate A and y -coordinate B .
- Carry out a rotation of each point by the required angle θ about the origin
- Translate all the points back to the rotation centre (A,B) by adding x -coordinate A and y -coordinate B .

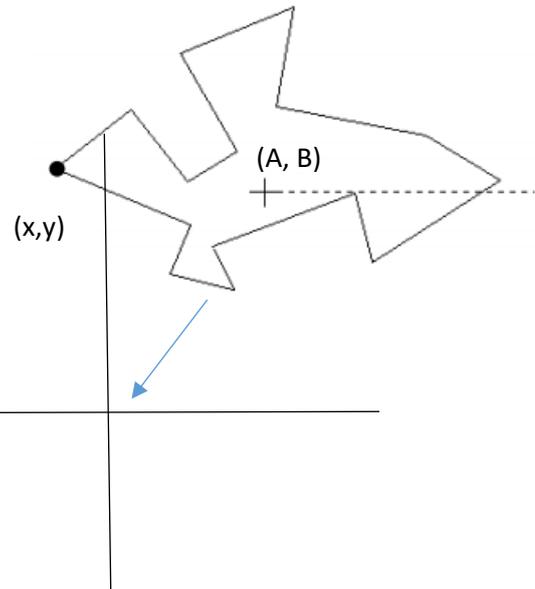
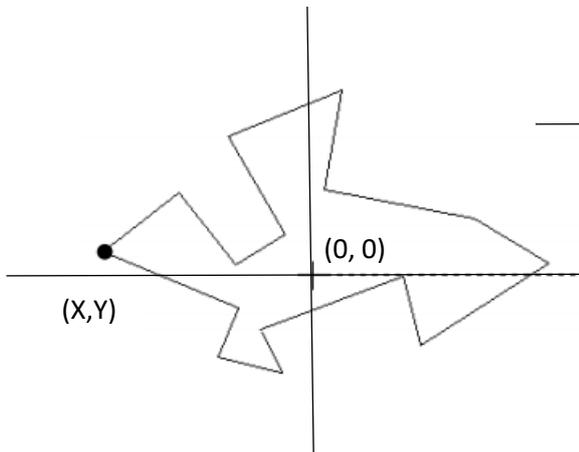
This sequence is illustrated in figure 172.

Step 1

Translate the shape from the rotation centre (A, B) to the origin using:

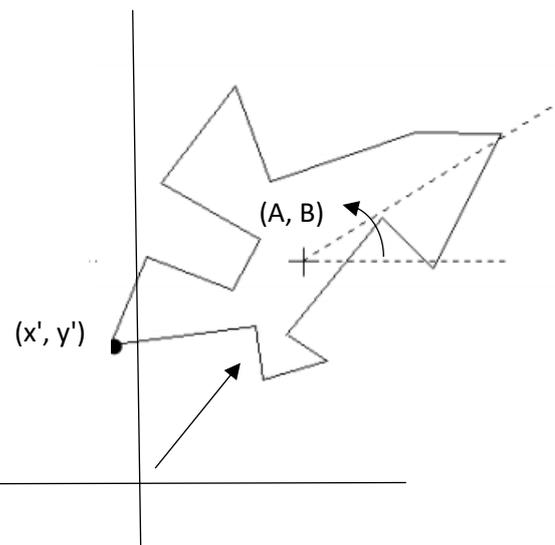
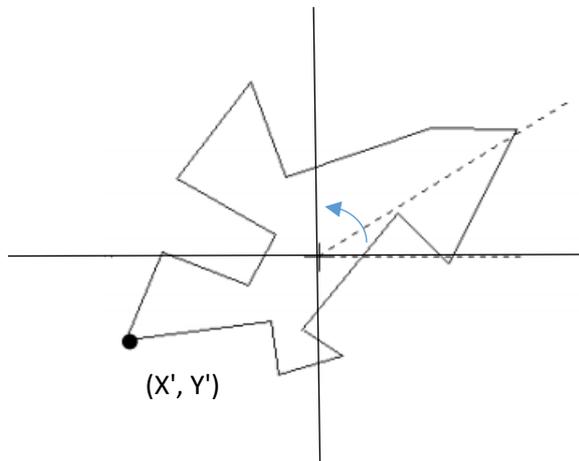
$$X = x - A$$

$$Y = y - B$$

**Step 2**

Rotate the shape at the origin using:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

**Step 3**

Translate the shape back to the rotation centre position using:

$$x' = X' + A$$

$$y' = Y' + B$$

Figure 172: Rotation of an object about a point which is not at the origin

The matrix containing the trigonometric functions:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

is known as the two dimensional rotation matrix. We can make use of this to develop more complex formulae for rotations in three dimensions.

Consider a cube. If we view the cube in the direction of the **z-axis**, we only see **face 1**:

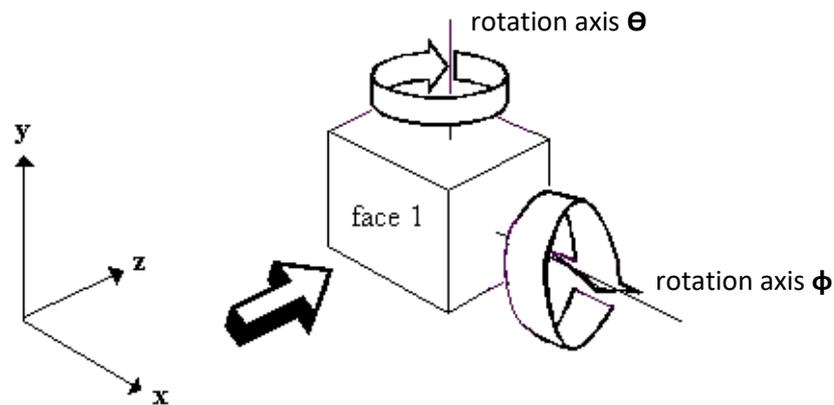


Figure 173: Rotation of a cube around two axes

To view the other faces, it will be necessary to apply a rotation to the cube. This is similar to the two dimensional rotation described previously. However, moving the cube into every possible position will require a combined rotation around **two** axes - we can call the two rotation angles **Θ** {theta} and **Φ** {phi}.

Consider first the rotation **Θ**. This takes place in the **x,z plane**:

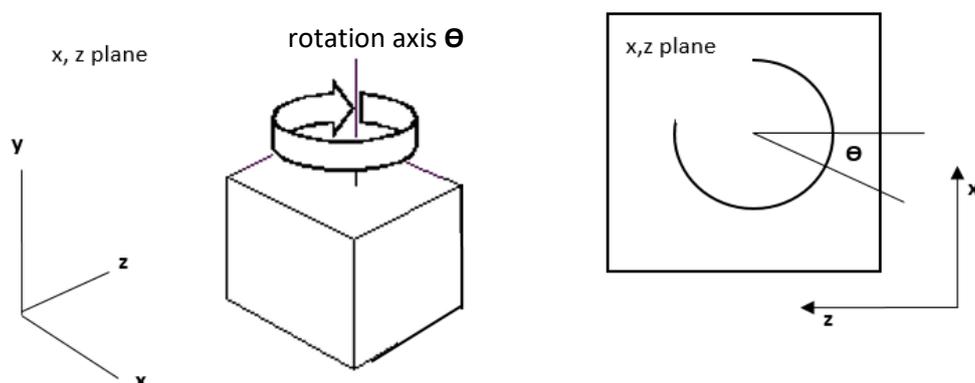


Figure 174: Rotation by an angle **Θ** in the x,z plane

Supposing that some point with the coordinates (\mathbf{x}, \mathbf{z}) is rotated by an **angle Θ** so that it ends up in position (\mathbf{X}, \mathbf{Z}) . The new coordinates of the point are given by the formulae:

$$X = x \cos \theta - z \sin \theta$$

$$Z = x \sin \theta + z \cos \theta$$

These are the same formulae as for two-dimensional rotation - we have just used z instead of y . The two equations can also be written in matrix form:

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

However, because we are working in three-dimensions, we mustn't ignore the y coordinate. This will be unaffected if the point rotates in the **\mathbf{x}, \mathbf{z} plane**, so the new coordinate Y will be the same as the old coordinate y . The full set of equations is therefore:

$$X = x \cos \theta - z \sin \theta$$

$$Y = y$$

$$Z = x \sin \theta + z \cos \theta$$

These three equations can also be written in matrix form:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The other rotation ϕ takes place in the **\mathbf{y}, \mathbf{z} plane**:

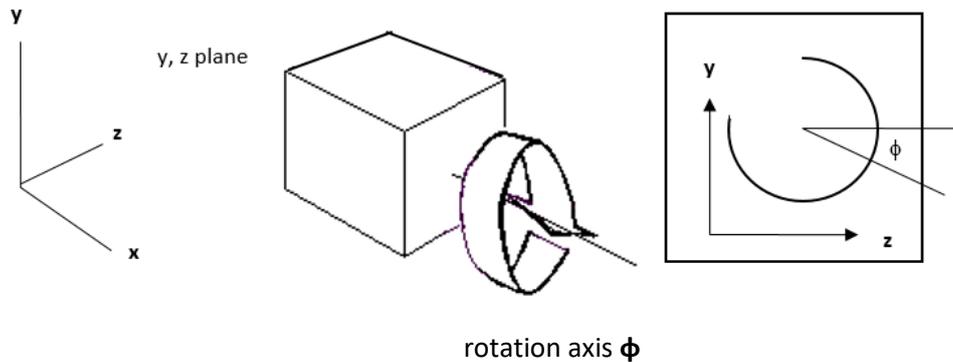


Figure 175: Rotation by an angle ϕ in the \mathbf{y}, \mathbf{z} plane

For rotations in the **\mathbf{y}, \mathbf{z} plane**, the x coordinate will be unaffected. The set of equations for a rotation are therefore:

$$X = x$$

$$Y = y \cos \theta - z \sin \theta$$

$$Z = y \sin \theta + z \cos \theta$$

These three equations can be written in matrix form:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We now arrive at the general case where the cube is rotated by both an **angle θ** and an **angle ϕ** . Together these allow us to turn the cube into any position we wish.

We can derive the three-dimensional rotation matrix by multiplying together the matrices for rotation in the **x, z** and **y, z** planes:

$$\begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \cdot \sin \phi & -\sin \theta \cdot \cos \phi \\ 0 & \cos \phi & -\sin \phi \\ \sin \theta & \cos \theta \cdot \sin \phi & \cos \theta \cdot \cos \phi \end{bmatrix}$$

This leads to the equations:

$$X = x \cdot \cos \theta - y \cdot \sin \theta \cdot \sin \phi - z \cdot \sin \theta \cdot \cos \phi$$

$$Y = y \cos \phi - z \sin \phi$$

$$Z = x \cdot \sin \theta + y \cdot \cos \theta \cdot \sin \phi + z \cdot \cos \theta \cdot \cos \phi$$

To see how this result can be used in computer graphics, we will look at a student project to display an animation of an excavator. The student began by creating a simple image of the vehicle by entering the corner coordinates of the series of surface patches which make up the solid shape:

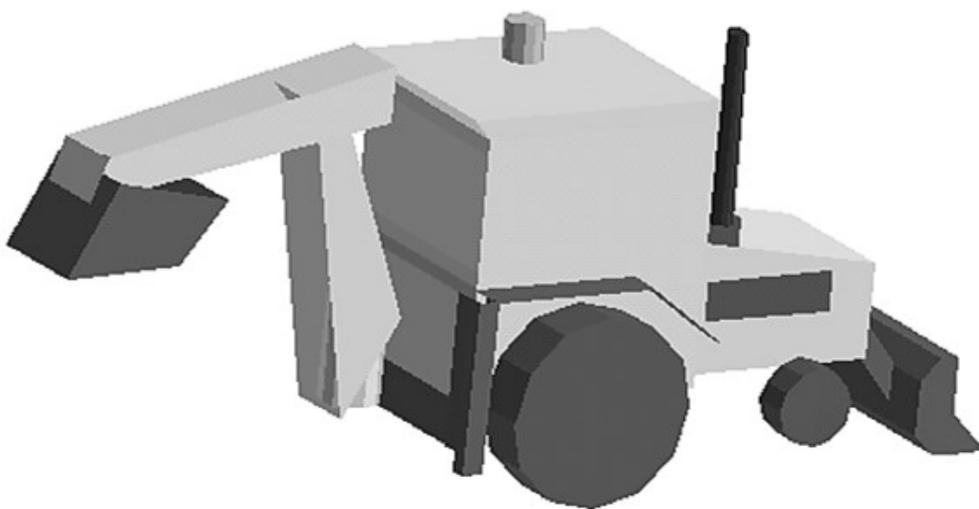


Figure 176: Basic solid model for the excavator

The rear excavator arm consists of three components which move independently. The boom may rotate horizontally to either side of the machine, and may elevate in a vertical plane. The dipper arm, attached to the top of the boom, may also elevate in a vertical plane. The bucket attached to the end of the dipper arm may rotate upwards and downwards. These motions are combined to produce a digging action.

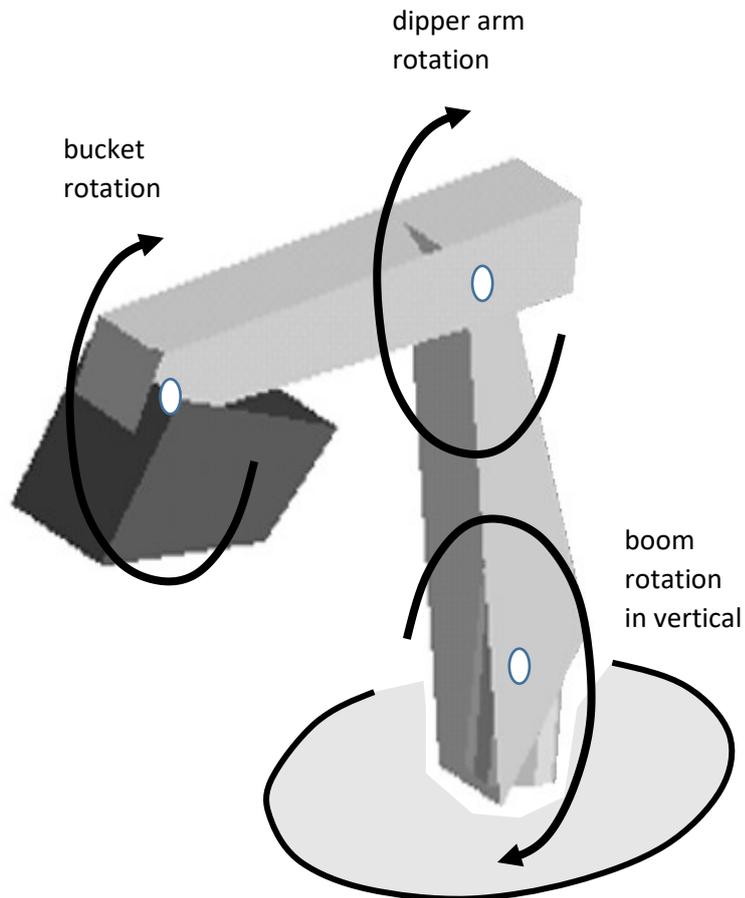


Figure 177: Independent motions of the rear excavator arm

In the computer animation, the user will be able to control the separate rotations by a set of virtual levers, in a similar way to real machine. The motions will then be shown in the graphics image of the excavator.

We will take the graphics origin as the intersection of the vertical and horizontal rotation axes for the boom. We will designate the horizontal rotation angle as θ and the vertical rotation angle as ϕ .

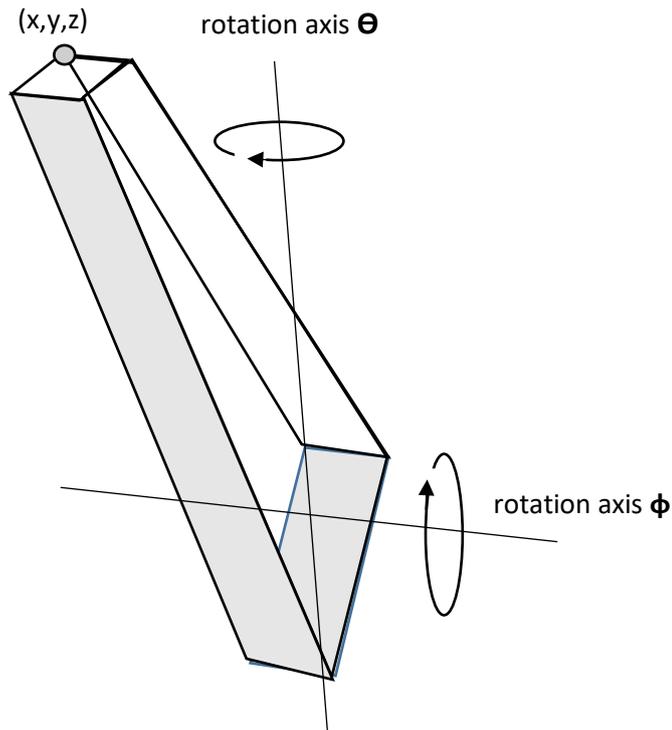


Figure 178: Angles of rotation of the excavator boom

If any point on the boom with current coordinates (x,y,z) is rotated by a horizontal angle Θ and a vertical angle ϕ , then the new coordinates of the point (X,Y,Z) will be given by:

$$X = x \cdot \cos \theta - y \cdot \sin \theta \cdot \sin \phi - z \cdot \sin \theta \cdot \cos \phi$$

$$Y = y \cos \phi - z \sin \phi$$

$$Z = x \cdot \sin \theta + y \cdot \cos \theta \cdot \sin \phi + z \cdot \cos \theta \cdot \cos \phi$$

We can now consider the motion of the dipper arm, which is pivoted at the end of the boom. Another axis of rotation is introduced, which we can call α

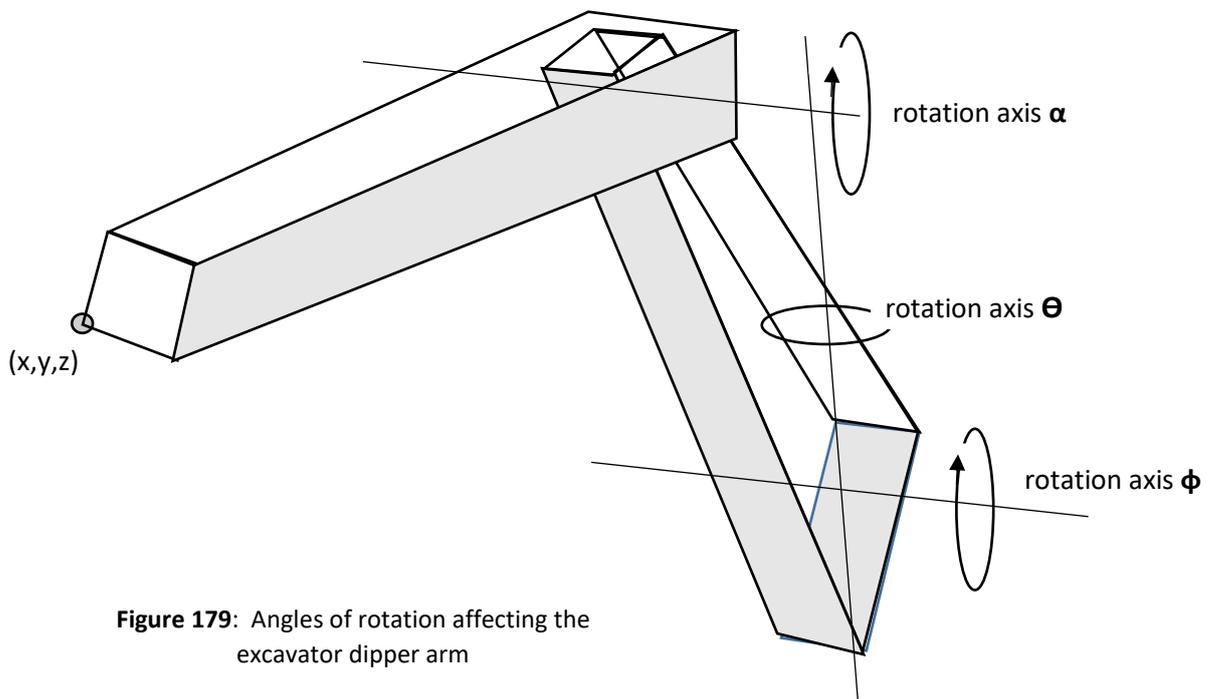


Figure 179: Angles of rotation affecting the excavator dipper arm

Suppose that a point (x,y,z) on the dipper arm moves through an angle α . The current position of the boom affects both the position of the rotation axis of the dipper arm, and the direction in which rotation of the dipper arm will take place. A strategy for rotating the dipper arm is:

- Translate each point on the dipper arm back to the origin. This can be done by subtracting the current x , y and z coordinates of the mid point of the rotation axis α
- Remove the effects of the current rotation of the boom. This can be done by applying a rotation of $-\Theta$ to each point on the dipper arm.
- Apply the vertical rotation α to the points of the dipper arm using:

$$X = x$$

$$Y = y \cos \alpha - z \sin \alpha$$

$$Z = y \sin \alpha + z \cos \alpha$$

- Re-apply the current rotation of the boom Θ to each point on the dipper arm.
- Translate each point on the dipper arm back to the current x , y and z position of the mid point of the rotation axis α

Motion of the bucket introduces yet another rotation axis, which we will call β

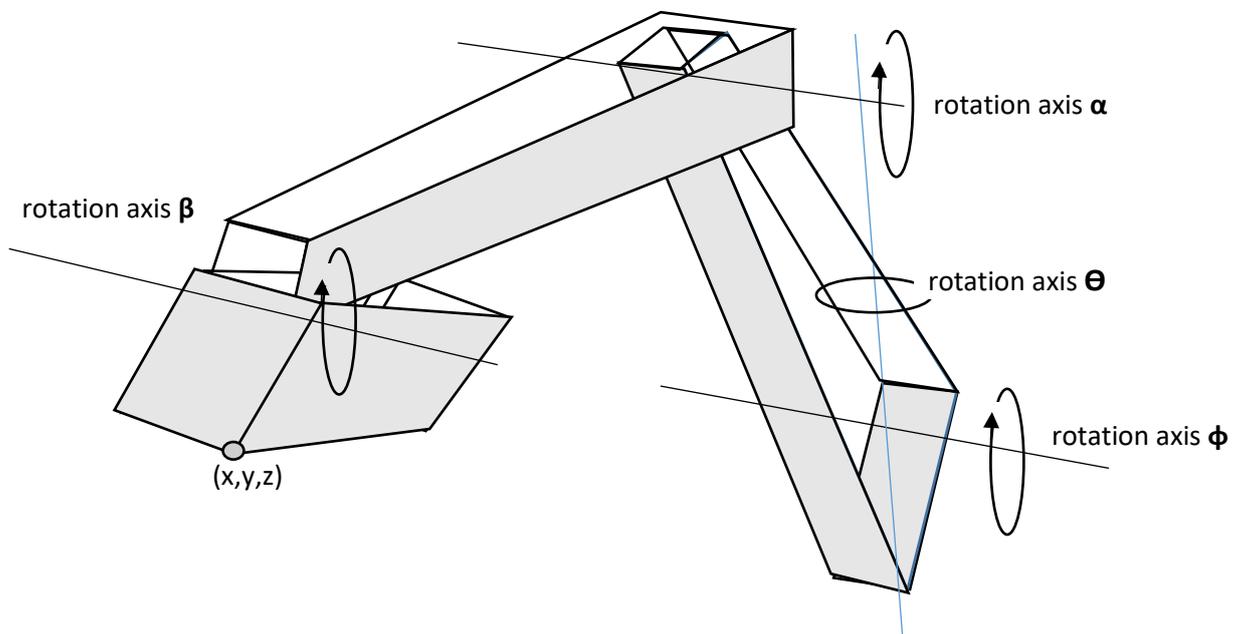


Figure 180: Angles of rotation affecting the excavator bucket

The strategy needed to apply a rotation of angle β to the bucket involves a series of stages:

- Translate each point on the bucket back to the origin. This is done by subtracting the current x , y and z coordinates of the mid point of the rotation axis β
- Remove the effects of the current rotation of the boom. This can be done by applying a rotation of $-\theta$ to each point on the bucket.
- Apply the vertical rotation β to the points of the bucket using:

$$X = x$$

$$Y = y \cos \beta - z \sin \beta$$

$$Z = y \sin \beta + z \cos \beta$$

- Re-apply the current rotation of the boom θ to each point on the bucket.
- Translate each point on the bucket back to the current x , y and z position of the mid point of the rotation axis β .

Summary

In this chapter we have looked at a range of mathematical techniques, from surveying land surfaces to representing components of buildings and machinery. Numeracy tasks involving shape and space can be relevant to a wide range of courses, and often provide interesting and motivating challenges for students.

Shape and space projects fit well into the scheme of broader numeracy training, incorporating: measurement, use of computer technology, problem solving and communication, and a requirement for a detailed subject knowledge of the vocational area.

Mathematical techniques required in shape and space tasks can extend well beyond the simple calculation of distances, areas and volumes involving regular shapes. Problems may require the application of some advanced geometrical and trigonometrical techniques. We will see in a later chapter that calculus may also play a part in shape and space calculations.