12 Calculus

Calculus is often seen as a complex and advanced area of mathematics, probably due to the complicated algebra required to solve calculus problems analytically. This is unfortunate, as calculus is based around some very simple and useful concepts which have immediate uses in a variety of subject areas. Complicated algebra can often be avoided by using simpler numerical methods which give acceptable accuracy. This then allows students to focus on interpreting the results produced.

Calculus can be divided into two main branches, **differentiation** and **integration**:

The objective of **differentiation** is to analyse a mathematical function to determine the rate at which the function is changing at different points. The rate of change is often expressed as the ratio of the sides of a triangle drawn as a tangent to the curve:

$$rate\ of\ change = \frac{dy}{dx}$$
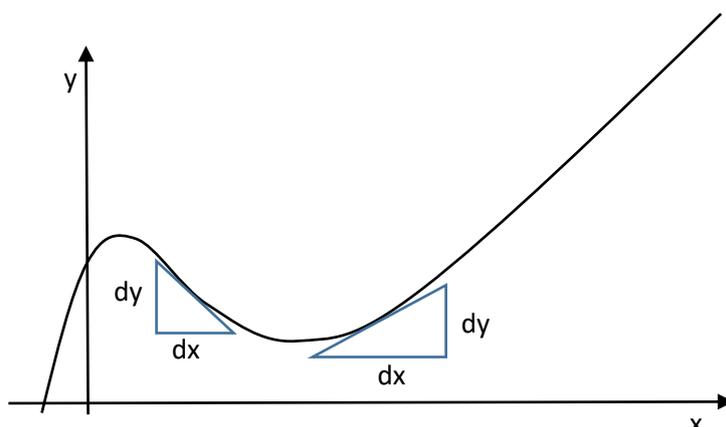
Figure 369: Gradient of a function

The relationship between a function value and its rate of change can be important in a variety of contexts. For example, the rate at which heat is lost from a room depends on the temperature of the room, and the rate at which new bacteria are produced depends on the current number of bacteria in the colony.

The approach taken in advanced-level mathematics courses is generally to use algebraic methods to determine a formula for the rate of change of a function. The rate of change is known as the **derivative**. For example, the function:

$$y = x^3$$

has the derivative:

$$\frac{dy}{dx} = 3x^2$$

$$function \quad y = x^3 \qquad\qquad derivative \quad \frac{dy}{dx} = 3x^2$$
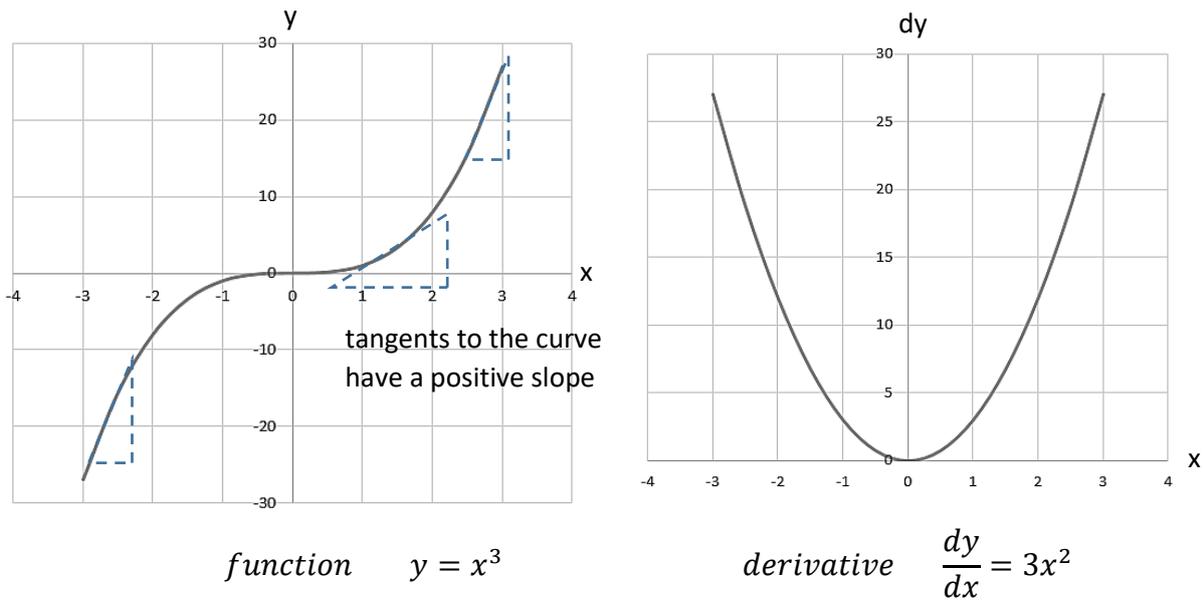
Figure 370: Graphs of y=x³ and its derivative

The function y=x³ has a double curve shape, rising steeply towards the origin, flattening out, then rising again increasingly steeply for positive x values. The graph of the derivative is a parabola, curving down to a minimum of zero at x=0.

By comparing the graphs above, we can see that the shape for the derivative is reasonable. If we were to examine points on the curve of y=x³ to assess the rate of change, all tangents would demonstrate an uphill positive slope with the exception of the point x=0 where the graph line is horizontal. Furthermore, the curve is steepest at x-values which are furthest away from zero in either the positive or negative direction. Looking now at the derivative curve, we see that all values are positive with the exception of the point x=0 where the rate of change is zero. The derivative shows higher values for the rate of change as the x-values move further away from zero in both the positive and negative directions.

The analytic method has been successful in determining the rate of change of the function y=x³, and can be used to find derivatives for a vast range of other functions. However, the algebra involved can be complex, and often adds little to the understanding of the problem which is being investigated. An alternative is to use **numerical** methods. We will explore this approach by again examining the function y=x³.

The simplest way to calculate the rate of change is **Euler's** method. We begin by producing a table of numerical values for the function y=x³, as shown in figure 371. Small steps for x of 0.1 have been chosen, so as to obtain reasonable accuracy. The change in y value for each increase in x value is used to determine the rate of change. For example, between the x positions of -3.0 and -2.9, we calculate that the y value changes from -27 to -24.39:

$$\frac{dy}{dx} = \frac{(-27) - (-24.39)}{(-3.0) - (-2.9)} = \frac{-2.61}{-0.1}$$

$$\frac{dy}{dx} = 26.1$$

	A	B	C	D	E	F	G	H	I	J	K
1	x	x³	gradient								
2	-3	-27	26.11								
3	-2.9	-24.389	24.37								
4	-2.8	-21.952	22.69								
5	-2.7	-19.683	21.07								
6	-2.6	-17.576	19.51								
7	-2.5	-15.625	18.01								
8	-2.4	-13.824	16.57								
9	-2.3	-12.167	15.19								
10	-2.2	-10.648	13.87								
11	-2.1	-9.261	12.61								
12	-2	-8	11.41								
13	-1.9	-6.859	10.27								
14	-1.8	-5.832	9.19								
15	-1.7	-4.913	8.17								
16	-1.6	-4.096	7.21								
17	-1.5	-3.375	6.31								
18	-1.4	-2.744	5.47								
19	-1.3	-2.197	4.69								
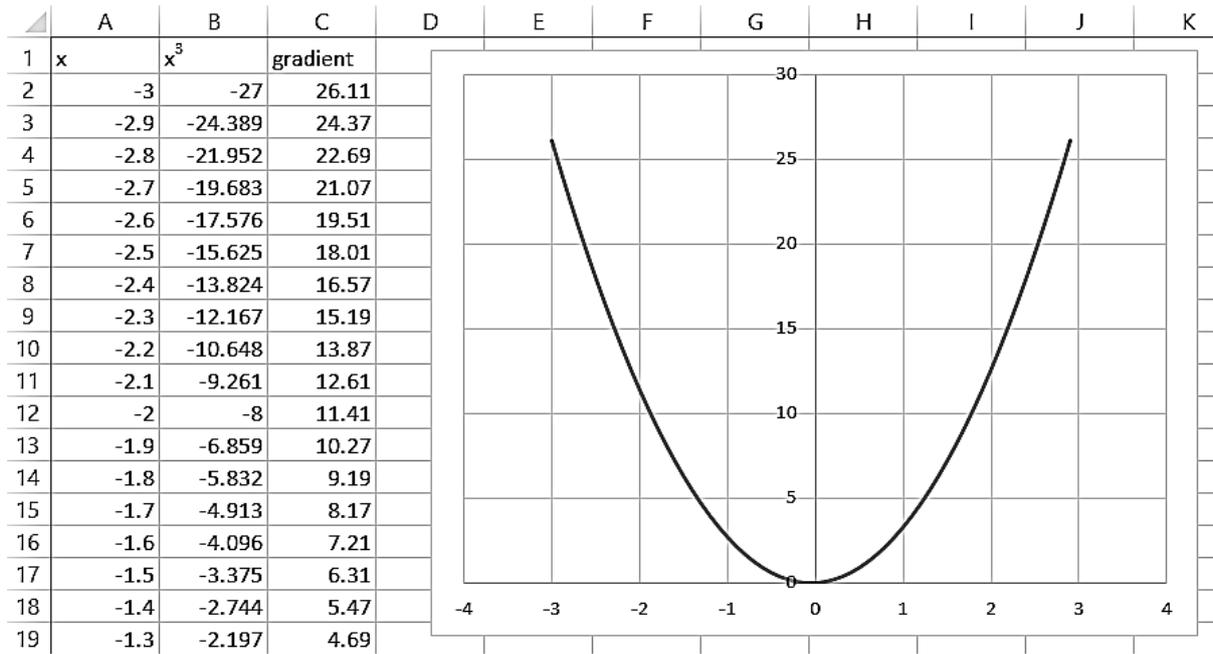
Figure 371: Determining the derivative of $y=x^3$ by Euler's method

By comparing figures 370 and 371, we see that the result obtained by the numerical method is very close to the analytical result, and would be satisfactory to use for most practical purposes.

Integration is the second major branch of calculus. We again start with a function, but we now determine the area underneath the graph, rather than its gradient. For example, the function:

$$y = x^3$$

has the integral:

$$\int y . dx = \frac{1}{4}x^4$$

triangle with an area of 20 units for comparison

$$function \quad y = x^3 \qquad\qquad integral \quad \int y . dx = \frac{1}{4}x^4$$
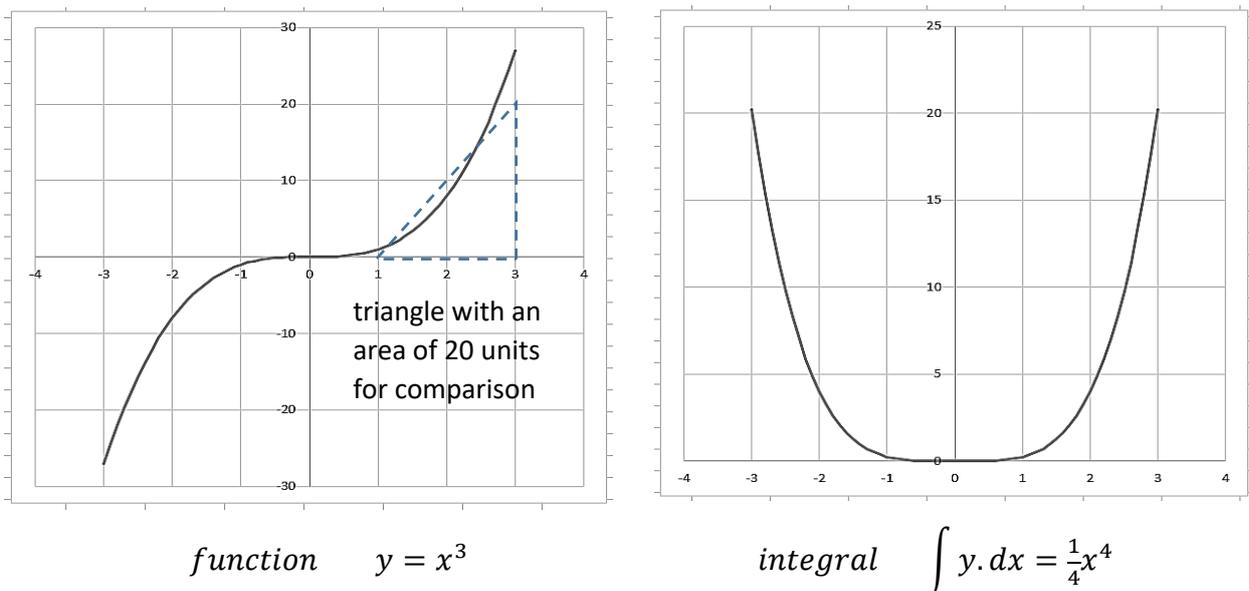
Figure 372: Graphs of $y=x^3$ and its integral

To find the area underneath the graph line of $y=x^3$ between any two selected x-values, we obtain the values of the integral at the two x points, then subtract one from the other. For example, to find the area under the graph between x=1 and x=3:

value of the integral at x = 1:   0.25

value of the integral at x = 3:   20.25

Subtracting:

20.25 – 0.25 = 20.00

This result seems reasonable when compared to a triangle of area 20 units which has been drawn next to the graph line.

The area under the graph of a function can be important in a variety of calculations. For example, the area under a velocity-time graph represents the distance travelled, and the area under a graph of electrical power in Watts against time in seconds represents electrical energy in Joules.

As in the case of finding derivatives, it is possible to use analytic methods to obtain the integrals of a huge range of functions. However, the algebra can again be complex and does not add much to the understanding of the task. Sufficiently accurate results can usually be obtained in a more straight-forward way by numerical methods. We will demonstrate the **trapezoidal method** by finding the area under the graph of $y=x^3$ between the points x=1 and x=3. This approach is illustrated in figure 373 below:

x	x^3	column area
1	1	0.08645
1.1	1.331	0.11655
1.2	1.728	0.15295
1.3	2.197	0.19625
1.4	2.744	0.24705
1.5	3.375	0.30595
1.6	4.096	0.37355
1.7	4.913	0.45045
1.8	5.832	0.53725
1.9	6.859	0.63455
2	8	0.74295
2.1	9.261	0.86305
2.2	10.648	0.99545
2.3	12.167	1.14075
2.4	13.824	1.29955
2.5	15.625	1.47245
2.6	17.576	1.66005
2.7	19.683	1.86295
2.8	21.952	2.08175
2.9	24.389	2.31705
3	27	2.56945
	total area	20.02
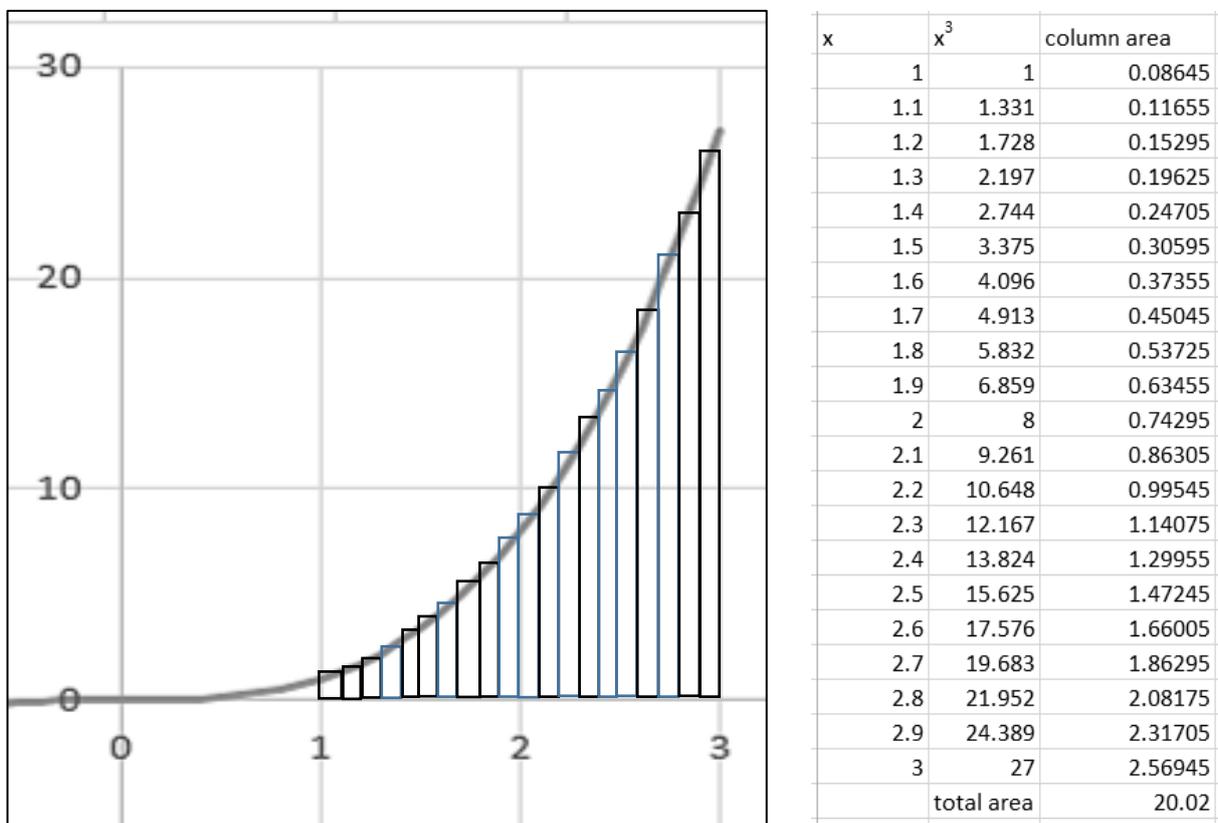
Figure 373: Use of the trapezoidal method to find the area below the graph of $y=x^3$

We begin by calculating values for the function y=x^3 at closely spaced points between x=1 and x=3. An interval of 0.1 has been chosen to provide reasonable accuracy. A series of rectangles are then constructed, and their areas found. The area of each rectangle is calculated as the average of the function values at the start and finish of the interval, multiplied by the interval width of 0.1.

$$A = \frac{(y1 + y2)}{2} \times 0.1$$
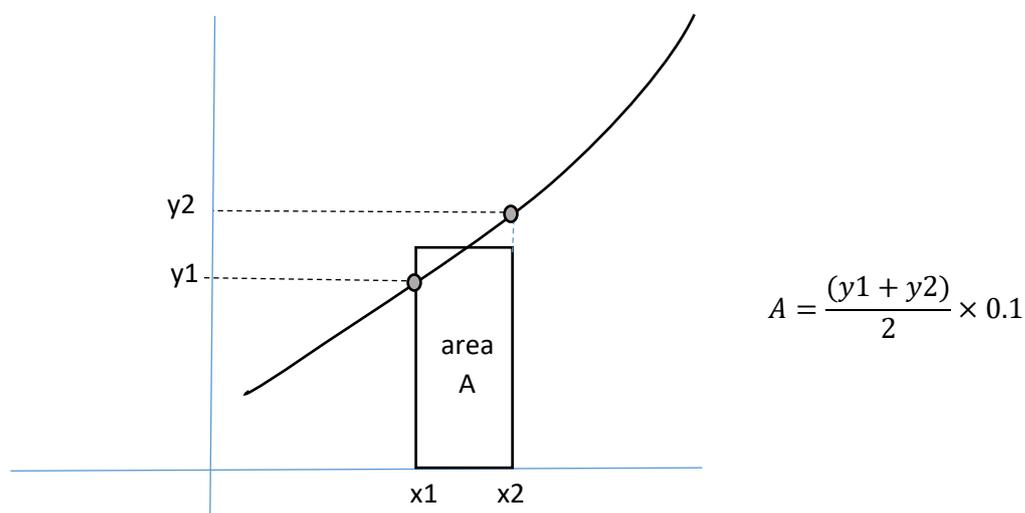
Figure 374: Estimating area below a graph by the trapezoidal method

The calculated areas for the set of columns are added to obtain a final estimate of the area beneath the function curve. We obtain a result of 20.02, which is sufficiently close to the accurate value of 20.00 for most practical purposes.

In the sections which follow, we examine a number of numeracy projects involving the use of calculus. Solutions will often be obtained by numerical methods, as these can be helpful to students in gaining a clearer understanding of the calculations required.

Accelerated motion for a roller coaster

Calculus techniques can be useful in analysing the motion of moving vehicles or machinery. In this section we will analyse data for the 'Crazy Caterpillar', a small roller coaster situated on Barmouth promenade. Although very tame in comparison to its larger cousins, the Crazy Caterpillar does demonstrate some important basic principles of mechanics.

The roller coaster has a figure-of-eight layout (figure 375), with the cars making two circuits at different levels. After leaving the boarding station, the train accelerates down a short drop then run around a curve to the base of the lift hill. The cars then ascend to the upper level at a height of four metres (figure 376).

After running around the upper curve and crossing the centre of the site, the cars loop back around another upper level curve. This brings the train to a final fast descent, in which the coaster reaches its maximum velocity (figure 377). After rounding the curve at the lower level, the cars return to the boarding point.

24m

9m
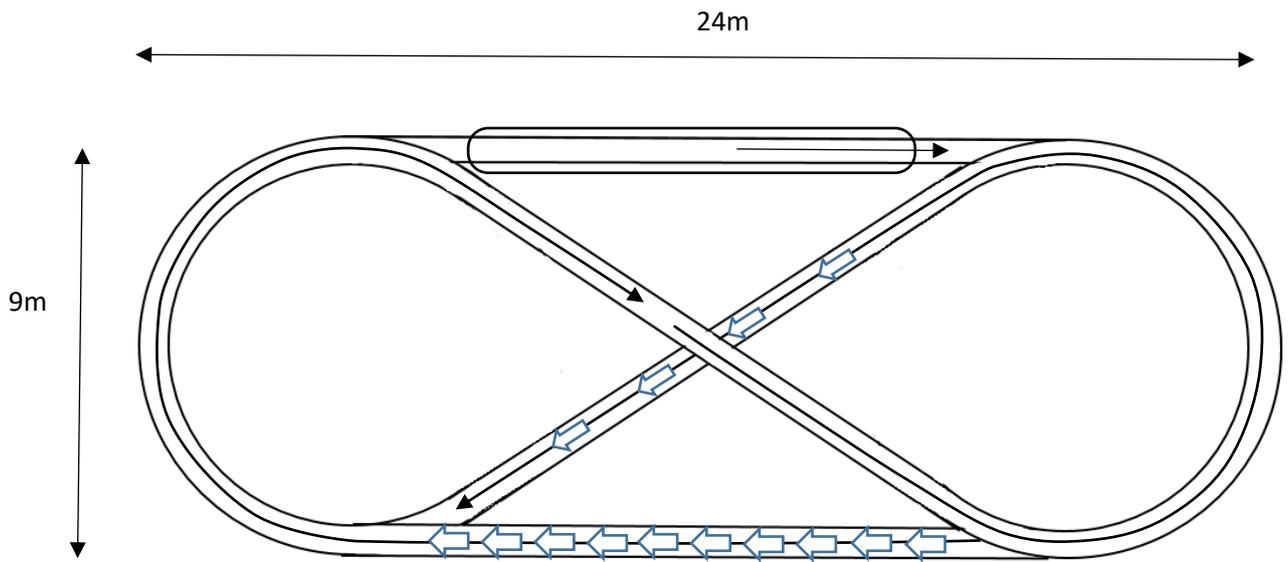
Figure 375: Track layout for the Crazy Caterpillar roller coaster

Figure 376: Cars on the lift hill and upper level track

Figure 377: Cars descending the incline from the upper level to the lower level

Our objective is to make an analysis of the motion of the roller coaster, in terms of the position, velocity and acceleration of the train of cars at different points around the track. The first step is to produce a profile diagram for the track. Estimates of distances and heights along a circuit of the track are shown in figure 378. The lift hill rises from point 3 to point 4, and the final fast descent occurs between points 6 and 7.
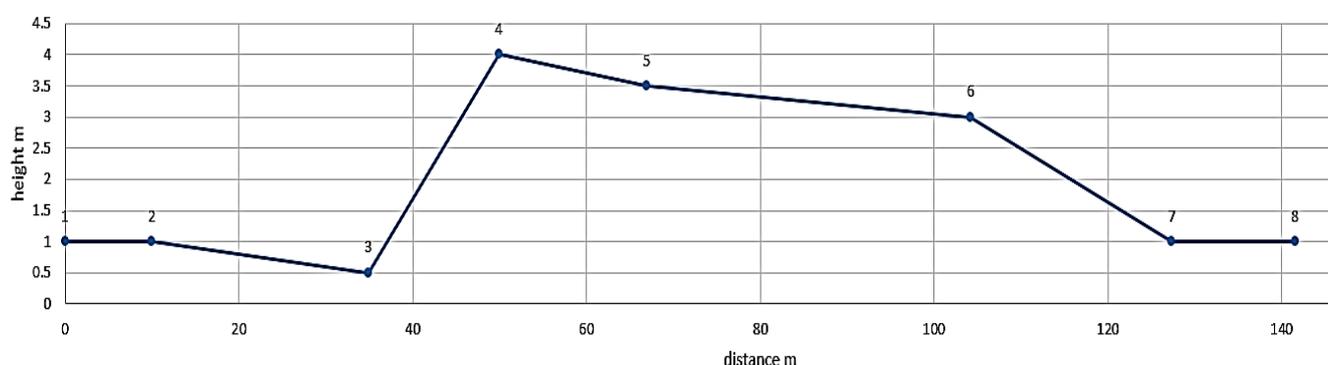
Figure 378: Height profile for a circuit of the roller coaster track

Times were recorded as the front of the train passed each of the numbered points during a circuit, then a distance-time plot was produced (figure 379).

point	distance m	time s
1	0	0
2	10	9
3	34.9	21
4	49.9	31
5	66.92	39
6	104.22	52
7	127.38	55
8	141.52	61

Figure 379:

Distance and time data for a circuit of the roller coaster track
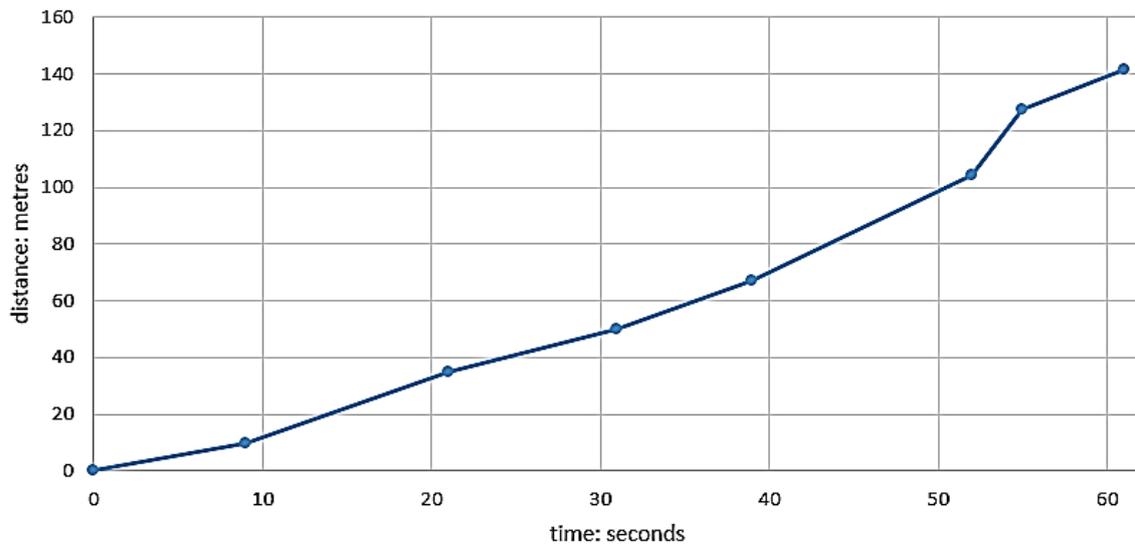
Figure 380: Distance-time curve for a circuit of the roller coaster track

Average velocity can now be calculated for each section of the track. Velocity is defined as change in position with time. Using the convention that the symbol **d** represents the change in a quantity:

$$velocity = \frac{ds}{dt}$$

where s is distance and t is time. Velocity can be estimated between the known points around the track using Euler's method. Values are tabulated below.

point	distance m	time s	d (distance)	d (time)	mean velocity
1	0	0			
			10	9	1.11
2	10	9			
			24.9	12	2.08
3	34.9	21			
			15	10	1.50
4	49.9	31			
			17.02	8	2.13
5	66.92	39			
			37.3	13	2.87
6	104.22	52			
			23.16	3	7.72
7	127.38	55			
			14.14	6	2.36
8	141.52	61			

Figure 381: Calculation of velocity data for the roller coaster

The velocity data has been plotted on a time graph in figure 382. At this stage, the points have been connected by straight lines. This is an over-simplified interpretation, and the graph will be revised later in the procedure.
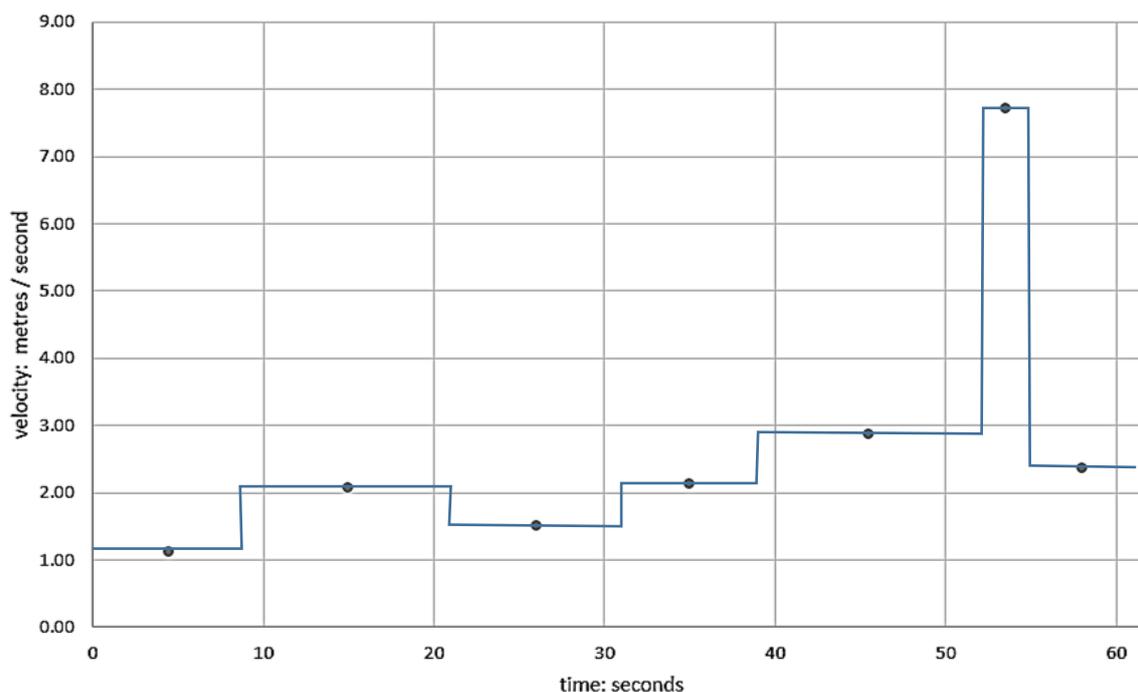
Figure 382: Velocity-time data for a circuit of the roller coaster track

Average acceleration can now be calculated for each section of the track. Acceleration is defined as change in velocity with time. Again using the convention that the symbol **d** represents the change in a quantity:

$$a = \frac{dv}{dt}$$

where a is acceleration, v is velocity and t is time. We might also describe acceleration as the rate of change of 'the rate of change of distance with time', giving the expressions:

$$a = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2s}{dt^2}$$

Euler's method can be used to estimate the acceleration between points on the track:

point	distance m	time s	mean velocity	mean time	d(velocity)	d(time)	mean acceleration
1	0	0					
			1.11	4.5			
2	10	9			0.96	10.50	0.092
			2.08	15			
3	34.9	21			-0.58	11.00	-0.052
			1.50	26			
4	49.9	31			0.63	9.00	0.070
			2.13	35			
5	66.92	39			0.74	10.50	0.071
			2.87	45.5			
6	104.22	52			4.85	8.00	0.606
			7.72	53.5			
7	127.38	55			-5.36	4.50	-1.192
			2.36	58			
8	141.52	61					

Figure 383: Calculation of acceleration data for the roller coaster
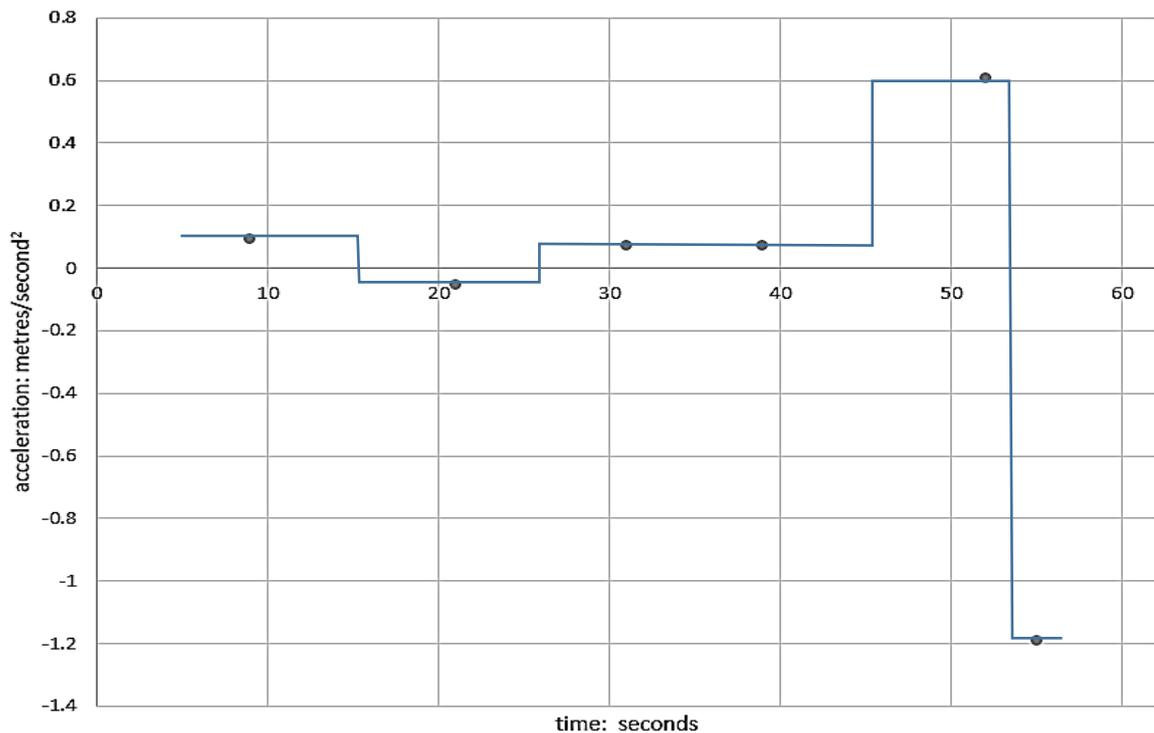
Figure 384: Acceleration-time data for a circuit of the roller coaster track

The initial plots of velocity and acceleration can now be refined to more closely represent the events taking place as a train of cars makes a circuit. The area beneath the graph of acceleration represents velocity, as given by the integral:

$$v = \int a.\,dt$$

Similarly, the area beneath the graph of velocity represents distance:

$$s = \int v.\,dt$$

Whilst the shapes of the velocity and acceleration curves can be changed, we should try not to alter the total areas beneath the curves. Final interpretations are given in figure 385 below:

- The train accelerates down a small slope after leaving the boarding station. The velocity increases.
- Deceleration occurs as the train approaches the base of the lift hill. The train then ascends at a constant velocity.
- At the top of the lift hill, the train is released down a gentle gradient and begins to accelerate.
- The steepest section of the track is reached, and acceleration increases. The cars reach their maximum velocity.
- At the base of the descent, the cars climb a short slope and round a sharp curve, causing rapid deceleration before the boarding station is reached at the end of the circuit.
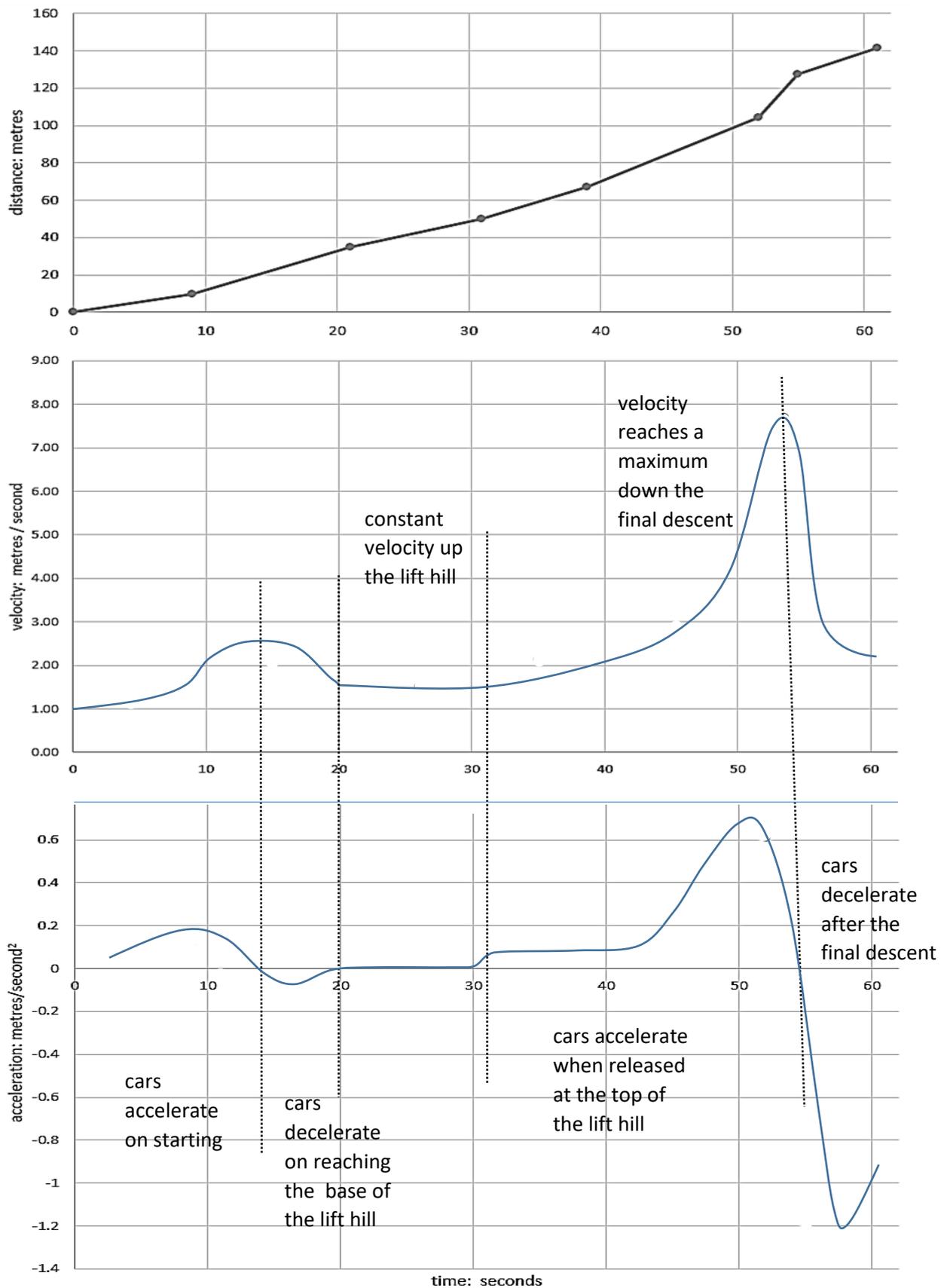
Figure 385: Completed analysis of the roller coaster motion

The final curve is taken at quite a high speed. Sharp and fast curves on roller coasters are often banked. There are several good reasons for this.

- When travelling around a curve, a centrifugal effect tends to impel the train straight ahead in a tangential direction. Centripetal force is needed in order to maintain the turn and hold the cars on the track. This force is provided by a horizonal reaction between the outer rail and the wheels. This can lead to increased and uneven wear on the track and the axle bearings of the cars. Banking the track reduces the amount of horizontal force which is applied to the car wheels.

- Riders on a horizontal track experience a centrifugal effect which throws them sideways as the train rounds a curve at speed. This can be uncomfortable. The ride feels smoother if the track is banked.

Ideally, a banking angle should be applied to the track which provides the exact turning force needed to counter the centrifugal effect of the curve. The cars will then round the curve with minimal sideways force against the wheels, and riders will not experience a sideways pull.

The angle of banking will depend on both the speed of the train and the radius of the curve. A steeper banking angle is needed for a higher speed and a tighter curve. Forces which need to be considered are shown in figure 386:
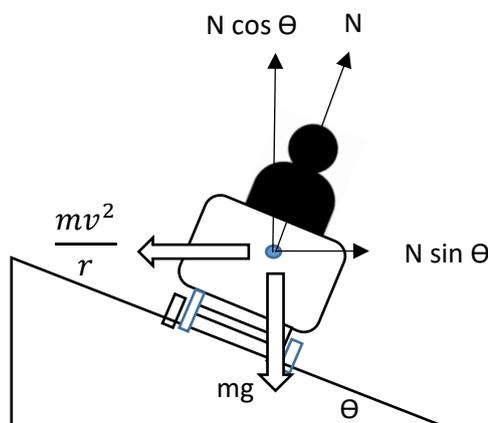
Figure 386:

forces acting on a roller coaster travelling around a banked curve

Two forces are acting on the car as it rounds a curve:

- The weight of the car, given by:
$$mg$$
where m is the mass of the car and g is the acceleration due to gravity of 9.8m/s^2.

- The centrifugal effect. This must be balanced by a centripetal force in order to keep the car on its circular path. Both the centrifugal effect and the centripetal force have magnitudes of:
$$\frac{mv^2}{r}$$
where m is again the mass of the car, v is its velocity in m/s^2, and r is the radius of the curve in metres.

If the car travels around the curve with exactly the correct velocity and banking angle so that there is no sideways force, all the force of the car will be downwards into the track. This will be balanced by a reaction force N which acts in an upwards direction perpendicular to the track.

The normal reaction force N can be split into two components:

(1) A vertical component which balances the weight of the car
$$N \cos \theta = mg$$
(2) A horizontal component which creates the centripetal force:
$$N \sin \theta = \frac{mv^2}{r}$$

Rearranging the first equation gives an expression for N in terms of the car weight and the angle of banking:

$$N = \frac{mg}{\cos \theta}$$

Substituting for N in the second equation gives:

$$\left(\frac{mg}{\cos \theta}\right) \sin \theta = \frac{mv^2}{r}$$

Since $\tan \theta = \sin \theta / \cos \theta$,

$$mg \, \tan \theta = \frac{mv^2}{r}$$

Cancelling m on both sides and rearranging gives:

$$\tan \theta = \frac{v^2}{rg}$$

We now have an equation which will give the banking angle θ required for a curve of radius r and a car velocity v.

The final curve of the roller coaster has a radius of 4.5 metres. The train is decelerating from 7.7 m/s to 2.3 m/s as it rounds this curve, so we can assume an average velocity of 5.0 m/s as the mid-section of the train passes through the curve. Substituting these values:

$$\tan \theta = \frac{5.0^2}{4.5 \times 9.8}$$

giving a result of 29° for the required angle of banking. In practice, the velocity will vary above and below 5.0 m/s, so the cars will need some frictional assistance to keep them on the track.

The exponential function

There are many cases in science where the **rate of change** of some material is proportional to the **quantity** of the material currently present. For example:

- Bacterial growth when nutrient supply is not limited. The rate at which new bacteria are added to the colony depends only on the number of bacteria reproducing.
- Radioactive decay, where the number of atoms decaying depends only on the amount of the radioactive isotope present.
- Discharge of an electronic capacitor, where the rate at which charge is leaving the capacitor depends on the amount of charge which is still held on the plates.

We will examine these processes in more detail shortly, but first we will consider the underlying mathematics of **exponential growth and decay**.

Suppose that a rate of change, such as the increase in the number of bacteria, is **equal** to the quantity itself. We can experiment with a spreadsheet to try to find a mathematical function to describe this relationship.

As a first guess, let us suppose that some time interval exists such that the number of bacteria will double during each of these intervals:

time	number of bacteria	
1	2	
2	4	
3	8	
4	16	
5	32	…. etc.

This relationship is described by the equation:

$$y = 2^x$$

where **y** is the number of bacteria at time **x**.

The rate of change in the number of bacteria can be determined numerically using Euler's method:

$$\frac{dy}{dx} = \frac{quantity\ at\ time\ (n+1) - quantity\ at\ time\ (n)}{time\ interval}$$

Graphs of quantity **y** and rate of change $\frac{dy}{dx}$ have been plotted in figure 387 using an x-interval of 0.1 to give a reasonable accuracy.

It is found that the value of $\frac{dy}{dx}$ is consistently less than the value of y for any position along the horizontal axis. This means that the function

$$y = 2^x$$

fails to correctly describe a situation where rate of change is **equal** to the quantity itself.

x	power(base, x)	gradient	mid point
1.0	2.000	1.435	1.05
1.1	2.144	1.538	1.15
1.2	2.297	1.649	1.25
1.3	2.462	1.767	1.35
1.4	2.639	1.894	1.45
1.5	2.828	2.030	1.55
1.6	3.031	2.176	1.65
1.7	3.249	2.332	1.75
1.8	3.482	2.499	1.85
1.9	3.732	2.679	1.95
2.0	4.000	2.871	2.05
2.1	4.287	3.077	2.15
2.2	4.595	3.298	2.25
2.3	4.925	3.535	2.35
2.4	5.278	3.788	2.45
2.5	5.657	4.060	2.55
2.6	6.063	4.352	2.65
2.7	6.498	4.664	2.75
2.8	6.964	4.999	2.85
2.9	7.464	5.357	2.95
3.0	8.000		
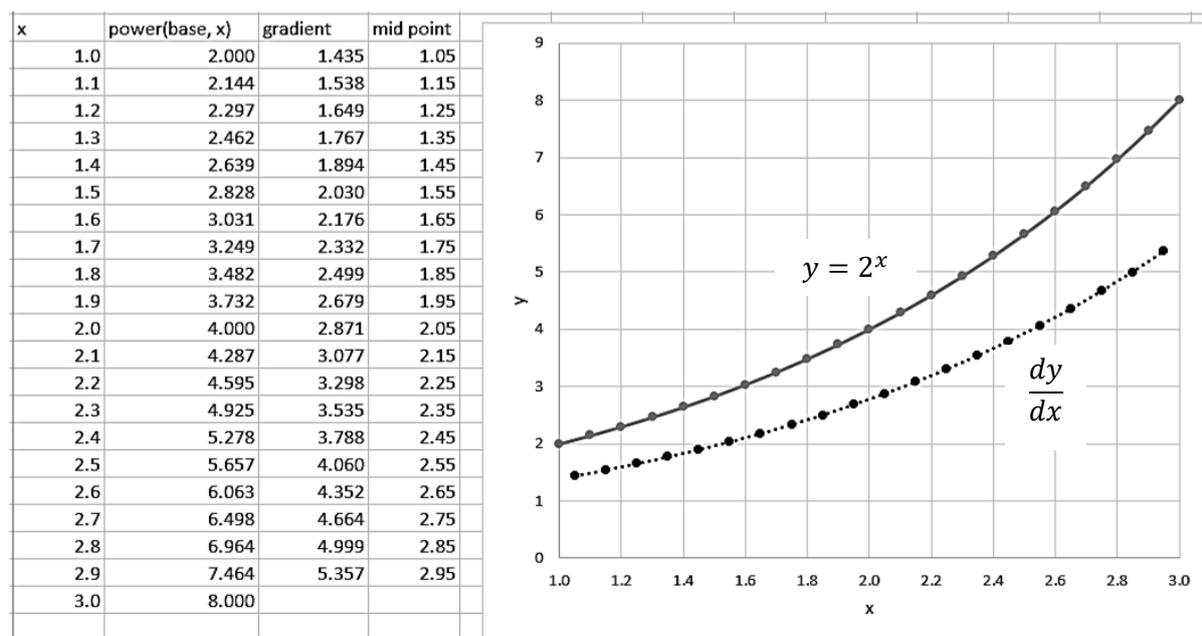
Figure 387: graph of y=2^x and its derivative

We need to find a function which will allow the numbers of bacteria to increase more rapidly. We can investigate the function:

$$y = 3^x$$

which assumes that the number of bacteria will triple during each successive time interval:

time	number of bacteria
1	3
2	9
3	27
4	81 …. etc.

Graphs of quantity **y** and rate of change $\frac{dy}{dx}$ for this function have been plotted in figure 388.

x	power(base, x)	gradient	mid point
1.0	3.000	3.484	1.05
1.1	3.348	3.888	1.15
1.2	3.737	4.340	1.25
1.3	4.171	4.844	1.35
1.4	4.656	5.406	1.45
1.5	5.196	6.034	1.55
1.6	5.800	6.735	1.65
1.7	6.473	7.517	1.75
1.8	7.225	8.390	1.85
1.9	8.064	9.364	1.95
2.0	9.000	10.451	2.05
2.1	10.045	11.665	2.15
2.2	11.212	13.019	2.25
2.3	12.514	14.531	2.35
2.4	13.967	16.218	2.45
2.5	15.588	18.102	2.55
2.6	17.399	20.204	2.65
2.7	19.419	22.550	2.75
2.8	21.674	25.169	2.85
2.9	24.191	28.091	2.95
3.0	27.000		

Figure 388: graph of y=3^x and its derivative

We now find that the value of $\frac{dy}{dx}$ is consistently greater than the value of y for any position along the horizontal axis. The function $y = 3^x$ again fails to describe a situation where rate of change is equal to the quantity itself, but is close to a correct result.

By further experimentation, we find that the quantity **y** and its derivative $\frac{dy}{dx}$ appear to be **equal** for all positions along the horizontal axis when a power of the number 2.7 is used.

x	power(base, x)	gradient	mid point
1.0	2.700	2.819	1.05
1.1	2.982	3.114	1.15
1.2	3.293	3.439	1.25
1.3	3.637	3.798	1.35
1.4	4.017	4.195	1.45
1.5	4.437	4.633	1.55
1.6	4.900	5.117	1.65
1.7	5.412	5.651	1.75
1.8	5.977	6.241	1.85
1.9	6.601	6.893	1.95
2.0	7.290	7.613	2.05
2.1	8.051	8.408	2.15
2.2	8.892	9.286	2.25
2.3	9.821	10.255	2.35
2.4	10.846	11.326	2.45
2.5	11.979	12.509	2.55
2.6	13.230	13.815	2.65
2.7	14.611	15.258	2.75
2.8	16.137	16.851	2.85
2.9	17.822	18.611	2.95
3.0	19.683		

Figure 389: graph of y=2.7^x and its derivative

This is a remarkable result whose importance cannot be over-stated. We have found a function:

$$y = \frac{dy}{dx} = 2.7^x$$

such that the amount of a quantity is equal to the rate at which that quantity is changing. The power which we found to be approximately 2.7 can be calculated accurately as:

$$2.71828183\ldots$$

This number has been defined as the constant **e**, named in honour of Euler:

$$If \quad y = e^x, \quad then \quad \frac{dy}{dx} = e^x$$

Problems where the **rate of change** of some material is proportional to the **quantity** of the material currently present are often solved analytically using the exponential function y = e^x, or some variant of this function.

Bacterial growth

We can now look in more detail at the growth of a colony of bacteria on a newly inoculated medium in a laboratory experiment. The process would typically involve four stages:

- **Lag phase**, in which the number of bacteria initially remains constant. During this time various processes are taking place, such as the synthesis of enzymes, which are necessary for growth to begin.
- **Exponential phase**, during which there are few inhibiting factors and bacteria can readily reproduce.
- **Stationary phase**, when growth is slowed and eventually halted. This may be due to exhaustion of the nutrient supply, a build-up of waste materials from cell metabolism, or simply a limitation of space for the colony to grow further.
- **Death**, in which the depletion of nutrients and build-up of waste materials cause a collapse in the bacteria population.

We will focus our attention on the middle two stages of the process, as initial exponential growth then slows to produce a steady final population. If we assume that the final population size is related to a constant **K**, and the number of bacteria present at time **t** is N_t, we can set up a recurrence relation:

$$N_{t+1} = \lambda N_t \left(1 - \frac{N_t}{K} \right)$$

This equation calculates the number of bacteria which will be present in the next time interval, N_{t+1}, based on the number which are present at the current time, N_t. The constant **λ** determines the rate of growth during a time interval: if **λ** is large then the bacteria will be reproducing more rapidly.

At the start of the exponential growth phase, the number of bacteria will be very low in comparison to the final population size. The term N_t / **K** is negligible, so the equation simplifies to:

$$N_{t+1} = \lambda N_t$$

This represents exponential growth, in which the increase in bacteria during each time step is directly proportional to the number of bacteria already present.

As the colony increases, the term N_t / **K** becomes larger, so that the multiplying term in the brackets is reduced to a fraction less than 1. At the point where:

$$\lambda \times \left(1 - \frac{N_t}{K} \right) = 1$$

λ is bigger than 1

$1 - \dfrac{N_t}{K}$ is smaller than 1

then:

$$N_{t+1} = N_t$$

and the population remains constant from one time step to the next.

This situation is illustrated in figure 390, using the values:

$$\lambda = 2$$
$$K = 1\,000\,000$$

Thus, at the equilibrium point,

$$2 \times \left(1 - \frac{N_t}{1\,000\,000}\right) = 1$$

$$\frac{N_t}{1\,000\,000} = 0.5$$

so N_t =500 000. The shape of graph produced is known as a **logistic** curve.

t	N								
	λ	2							
	K	100000							
1	4								
2	7.99968								
3	15.99808								
4	31.99104								
5	63.96161								
6	127.8414								
7	255.3559								
8	509.4078								
9	1013.626								
10	2006.702								
11	3932.868								
12	7556.387								
13	13970.79								
14	24037.93								
15	36519.41								
16	46365.48								
17	49735.8								
18	49998.6								
19	50000								
20	50000								
21	50000								
22	50000								
23	50000								
24	50000								
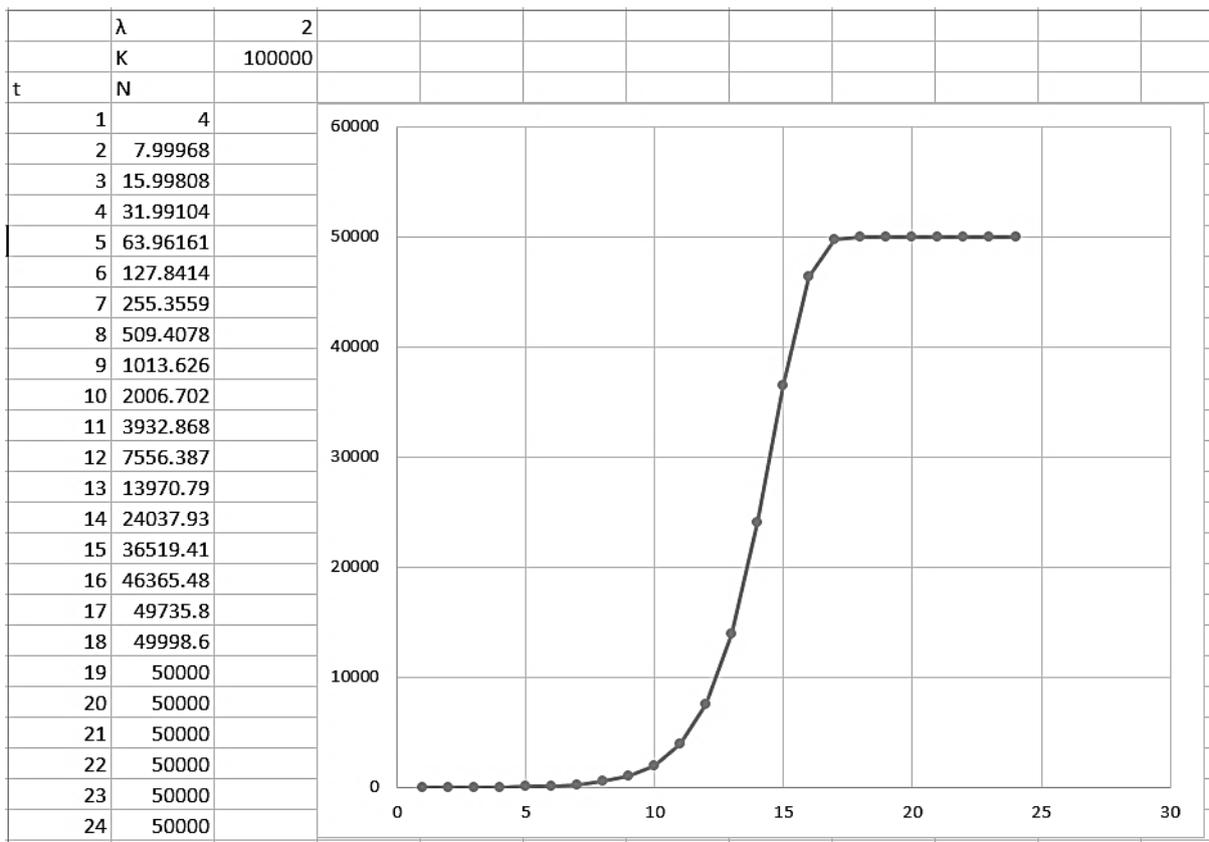
Figure 390: logistic curve for growth of a bacteria population using a numerical solution

We have examined the logistic function using a numerical approach, but an analytical solution by algebra is also possible. A starting point is the equation:

$$\frac{dN}{dt} = \lambda N \left(1 - \frac{N}{K}\right)$$

which states that the rate of increase in the bacteria population is proportional to the current number of bacteria multiplied by the bracket term:

$$\left(1 - \frac{N}{K}\right)$$

which progressively reduces as the population approaches its limiting value.

By means of calculus techniques which are beyond the scope of this book, but can be found in advanced level textbooks of mathematics, the logistic equation can be integrated to give:

$$N = \frac{Ce^{\lambda t}}{1 + \frac{C}{K}e^{\lambda t}}$$

This uses exponential functions to calculate the number of bacteria **N** which would be present in the population at any time **t**. **C** is another constant whose value is found during the integration, and depends on the initial and final numbers of bacteria in the population.

An example graph plotted with this equation is shown in figure 391 below:

C	50
K	5000
λ	4

t	N
0	50
0.1	73
0.2	109
0.3	161
0.4	236
0.5	344
0.6	496
0.7	706
0.8	985
0.9	1340
1	1766
1.1	2244
1.2	2743
1.3	3222
1.4	3650
1.5	4007
1.6	4288
1.7	4499
1.8	4653
1.9	4762
2	4838
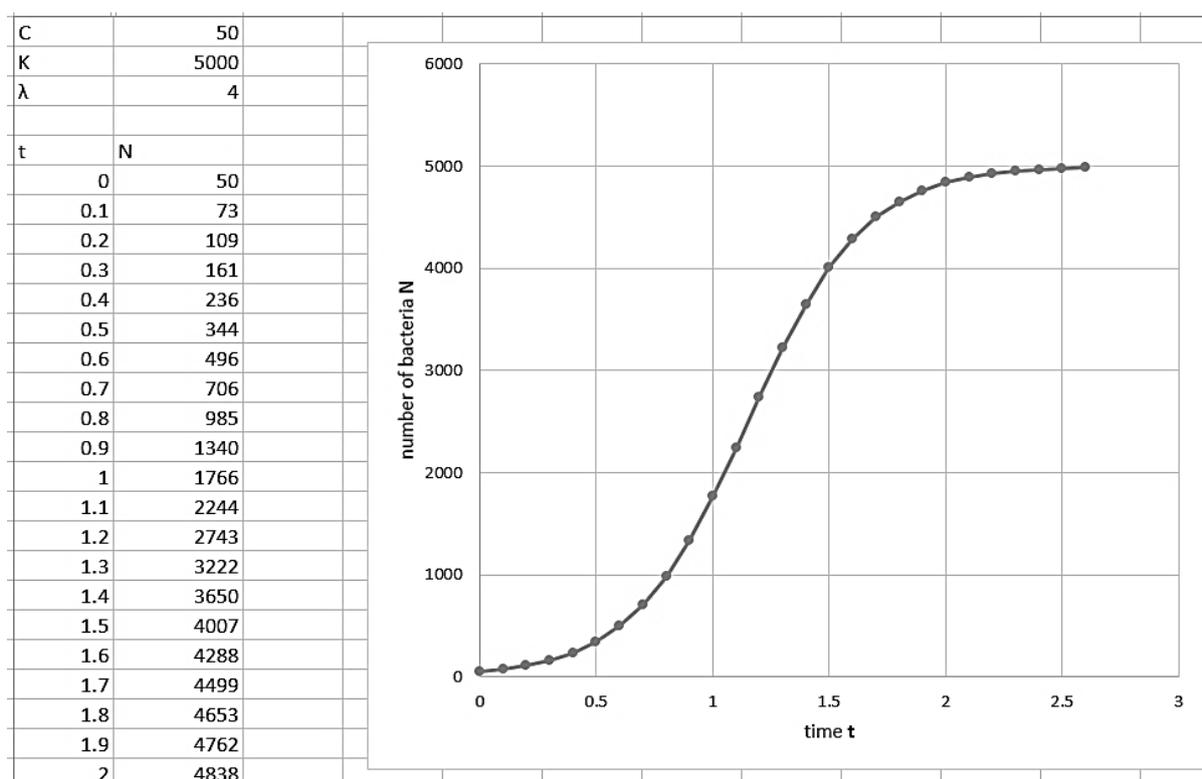
Figure 391: logistic curve for growth of a bacteria population using the analytic solution

Carbon dating

Carbon dating has become an important archaeological technique for determining the age of skeletons, or artefacts made from plant or animal materials such as wooden furniture or documents written on animal skin. It can also be used to determine the age of plant and animal materials preserved in relatively young geological formations such as sediments from the Ice Age.

Carbon exists naturally as two isotopes: ^{12}C which is stable, and ^{14}C which is unstable and undergoes radioactive decay to ^{12}C. The ratio of ^{12}C to ^{14}C in atmospheric carbon dioxide is close to constant. This same carbon isotope ratio is found in the organic compounds of living organisms, produced directly by photosynthesis or ingested as food substances.

After the death of the organism, no further transfer of carbon from the environment occurs. Atoms of ^{14}C decay to ^{12}C at a known rate. The number of years since the death of the organism can therefore be calculated by comparing the current carbon isotope ratio of the preserved material to the carbon isotope ratio at the time that the organism was living.

The **half-life** for the ^{14}C decay process is known to be 5,700 years. After this period, the original quantity of ^{14}C would be halved. After a further half-life of 5,700 years, the quantity of ^{14}C would be halved again. We should note that the rate of change in the amount of ^{14}C at any time is proportional to the amount of ^{14}C still present, so:

$$\frac{dy}{dt} = -\lambda y$$

where **y** represents the amount of ^{14}C and λ is a constant representing the rate of decay.

The reduction in the ^{14}C isotope content of a sample over time can be calculated numerically using a spreadsheet, as in figure 392 below.

half-lives	years	^{14}C isotope
	half life	5700
0	0	1
1	5700	0.5
2	11400	0.25
3	17100	0.125
4	22800	0.0625
5	28500	0.03125
6	34200	0.015625
7	39900	0.0078125
8	45600	0.00390625
9	51300	0.00195313
10	57000	0.00097656
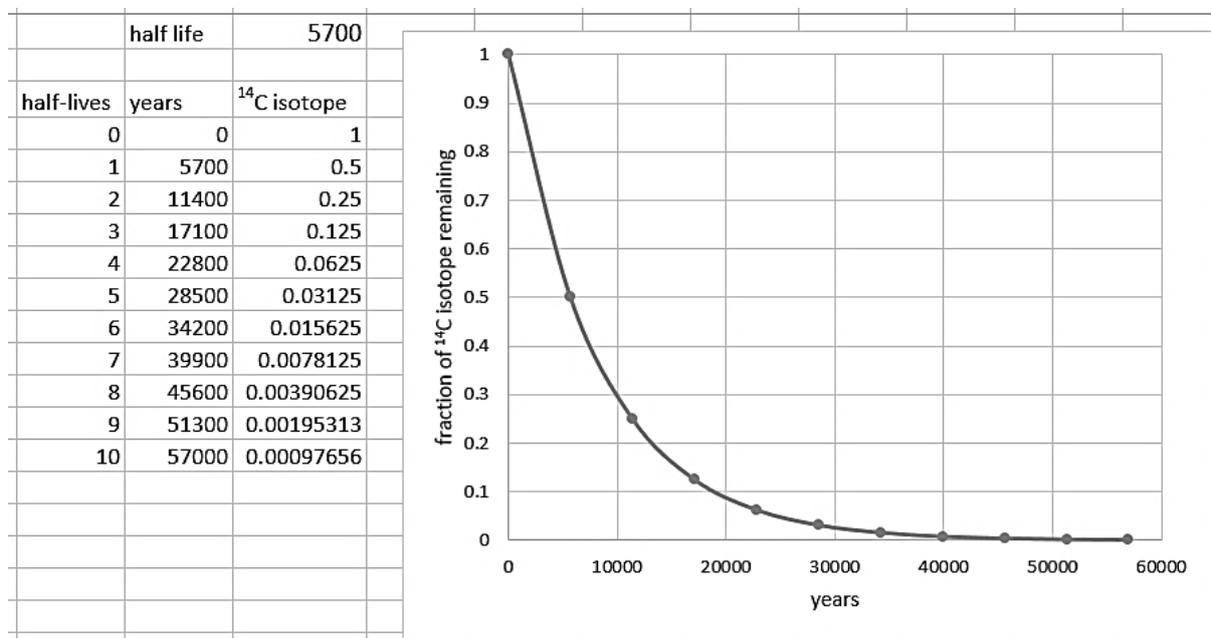
Figure 392: negative exponential curve for the radioactive decay of ^{14}C

Alternatively, algebraic techniques can be used to obtain the formula:

$$N(t) \ = \ N_0 e^{-\lambda t}$$

where **N(t)** is the number of grams of ^{14}C remaining in the sample at time **t**, **N₀** is the original number of grams of ^{14}C in the living material, and λ is a constant which can be calculated from the half-life.

It is valuable for students to see how this equation is derived, so that they gain an appreciation of the way that analytical methods are used in calculus. We begin with a statement that the rate of decay of ^{14}C atoms is proportional to the number present:

$$\frac{dN}{dt} = -\lambda \, N$$

Terms involving N are moved to the left of the equals sign. This is known as separation of variables:

$$\frac{dN}{N} = -\lambda \, dt$$

We then set up integrals on each side of the equation. Since λ is a constant, it can be left outside the integral:

$$\int \frac{1}{N} dN = - \lambda \int dt$$

The integral of $^1/_N$ with respect to **N** can be found in tables of standard integrals, and has the value log_e **N**. The right hand integral with respect to **t** is empty, so takes the value **t.** This gives:

$$log_e N = -\lambda t + C$$

Notice that a constant has been added. This must always be done when integration is carried out. We now need to find the value of this constant. We see that the number of ^{14}C atoms at time **t** = 0 is N_0. Substituting in the previous equation:

$$log_e N_0 = C$$

We can then substitute the value $log_e N_0$ for **C**:

$$log_e N = -\lambda t + log_e N_0$$

$$log_e N - \ log_e N_0 \ = -\lambda t$$

Using the property of logarithms that **log A – log B = log (A/B)**, we obtain:

$$log_e \left(\frac{N}{N_0} \right) = -\lambda t$$

We now raise each side of the equation to the power of **e**. This has the effect of reversing and removing the logarithm function, giving:

$$\frac{N}{N_0} = e^{-\lambda t}$$

This equation can be rearranged to give the required solution:

$$N = N_0 \, e^{-\lambda t}$$

For practical application of the carbon dating method, it is necessary to know the ^{14}C to ^{12}C ratio for the organism at its time of death, and also to be able to measure the current ^{14}C to ^{12}C ratio accurately. Both of these requirements are subject to some error, so the final calculated age is usually stated to be within a range of possible dates.

Our students have had interesting practical experience of the use of carbon dating during a geological and ecological investigation of a large area of peat blanket bog in the mountainous source area of the River Mawddach in Snowdonia. In places, the peat has been subject to erosion, exposing a layer of tree roots at the base.

Above: eroded peat deposits.

Left: layer of tree roots overlying fine gravel at the base of the peat.

Right: detail of tree roots embedded in the peat.

Figure 393: layer of tree roots at the base of peat deposits, Waen y Griafolen, North Wales

A sample from the tree root layer was submitted to the Oxford University Radiocarbon Accelerator Unit for dating. The result obtained was an age of 8,905 ± 45 years before the reference year AD 1950. This date represents the earliest return of forests to the uplands of western Britain following the end of the Ice Age (Bellamy, 1986). The roots are likely to be alder, which is known to have been present in Snowdonia at a time of 8,700 b.p. (Chambers and Price, 1985). At a slightly higher horizon within the peat are found water-transported birch branches. We have deduced that the blanket bog developed on the site of an extensive shallow lake overgrown by wet woodland. Peat accumulation appears to have continued with little interruption for the past 9,000 years, with mosses, heathers and other dwarf shrubs as the principal vegetation.

Capacitor charging and discharging

In Chapter 6, Measurement, we looked at a workshop experiment in electronics to produce a full-wave rectifier circuit. This included damping of the output voltage by capacitor, leading to a ripple effect when viewed on an oscilloscope screen:
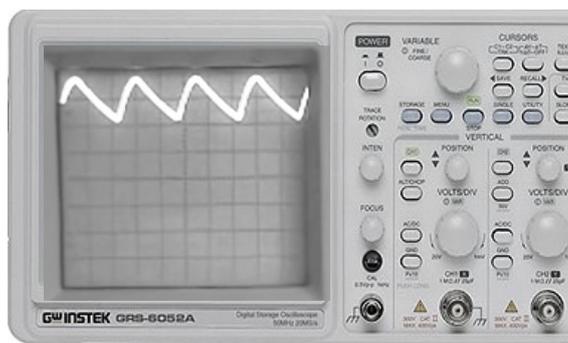
Figure 394:

oscilloscope displaying the output voltage from the full-wave rectifier circuit

We will now set up a spreadsheet model to simulate the output for this circuit. We begin with a bridge rectifier producing a sine wave output of positive peaks:
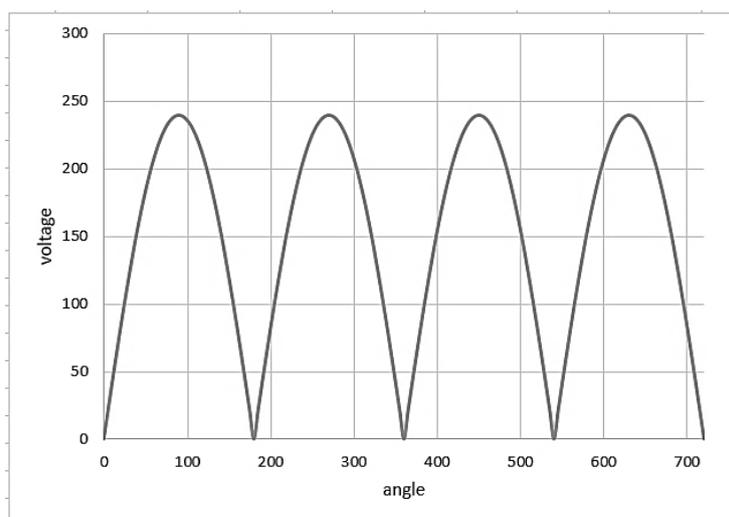
Figure 395:

positive sine wave output from the bridge rectifier

During the period when voltage is near its peak, the electrical charge in the smoothing capacitor can be replenished. This charge is then available to augment the output of the

circuit when the voltage falls. However, the voltage output from the capacitor itself will fall as charge is lost. The drop in capacitor voltage with time **t** can be specified by a negative exponential equation similar in pattern to the radioactive decay equation:

$$V(t) = V_0\, e^{-(t/RC)}$$

V(t) is voltage at time **t**, **V₀** is the voltage when the capacitor is fully charged, **R** is the circuit resistance, and **C** is the capacitance. We can deduce that the power of **e** will be larger, and voltage will therefore fall more quickly, if the resistance or the capacitance in the circuit is low.   A spreadsheet to simulate the rectifier output can be set up using the following steps:

- Constants are assigned to specify the peak voltage **V₀** of 240V, a resistance **R** with a typical value of 1000 Ω, and a capacitance **C** of 0.0001 F. Voltage angles **Ө** for the alternating current are set up at 5⁰ intervals from 0⁰ to 720⁰.
- Sine wave output is calculated using:
$$V = V_0 \sin\theta$$
- Sine waves are rectified to produce only positive peaks by means of the spreadsheet function ABS( ), which converts any negative values to their positive equivalent.
- An angle **ϕ** is defined which counts upwards in the same way as **Ө,** but is reset to zero at each sine wave peak. This angle is calculated from **Ө** using a MOD spreadsheet function:
$$\varphi = MOD((\Theta+90),\ 180)$$
- The angles **ϕ** are converted to times **t** in seconds, based on an alternating current frequency of 50Hz. Each cycle of 360⁰ will represent 1/50ᵗʰ of a second.
- Capacitor output voltage is then calculated using the equation:
$$V(t) = V_0\, e^{-(t/RC)}$$
- Finally, the spreadsheet **IF** function is used to compare the rectified sine wave and capacitor voltages, then select the higher of the two for output to the graph.

			R		1000	C		0.0001	
amplitude	240								
				capacitor					
angle	sine wave	rectified	t	voltage	output				
0	0	0	0.005	228.2951	228.2951				
5	20.91738	20.91738	0.005278	227.6618	227.6618				
10	41.67556	41.67556	0.005556	227.0303	227.0303				
15	62.11657	62.11657	0.005833	226.4005	226.4005				
20	82.08483	82.08483	0.006111	225.7725	225.7725				
25	101.4284	101.4284	0.006389	225.1462	225.1462				
30	120	120	0.006667	224.5217	224.5217				
35	137.6583	137.6583	0.006944	223.8989	223.8989				
40	154.269	154.269	0.007222	223.2778	223.2778				
45	169.7056	169.7056	0.0075	222.6584	222.6584				
50	183.8507	183.8507	0.007778	222.0408	222.0408				
55	196.5965	196.5965	0.008056	221.4249	221.4249				
60	207.8461	207.8461	0.008333	220.8107	220.8107				
65	217.5139	217.5139	0.008611	220.1981	220.1981				
70	225.5262	225.5262	0.008889	219.5873	225.5262				
75	231.8222	231.8222	0.009167	218.9782	231.8222				
80	236.3539	236.3539	0.009444	218.3708	236.3539				
85	239.0867	239.0867	0.009722	217.765	239.0867				
90	240	240	0	240	240				
95	239.0867	239.0867	0.000278	239.3343	239.3343				
100	236.3539	236.3539	0.000556	238.6704	238.6704				

Figure 396: spreadsheet simulation of the full-wave rectifier circuit

The completed spreadsheet for simulating the rectifier circuit is shown in figure 396 above. The initial values set for the resistance and capacitance of the circuit can now be varied to investigate the effects on circuit output. It is seen that the voltage ripple can be effectively removed by choice of a sufficiently large capacitor.
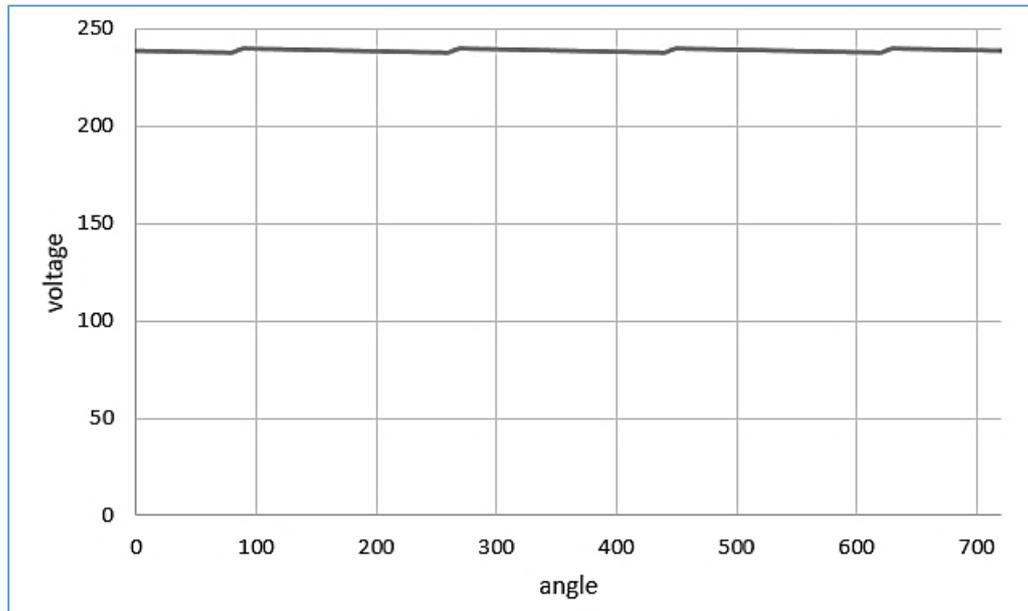
Figure 397: simulation of the full-wave rectifier circuit with large capacitance

Solid Shapes

An important use of integration is in finding the volumes and surface areas of solid shapes.

As an example, a geometrical problem has been used to introduce engineering students to calculus at the start of their course. Students are asked to estimate the volume of the centre cone of a jet engine air intake fan (figure 398):

Figure 398:

aircraft engine showing the air intake fan

The conical central section of the fan has a depth of 0.5m. The profile of the cone follows the mathematical function:

$$y = \sqrt{2x}$$

Students were told that the volume of the cone needed to be found, so that its mass could be calculated for modelling the rotation of the fan.
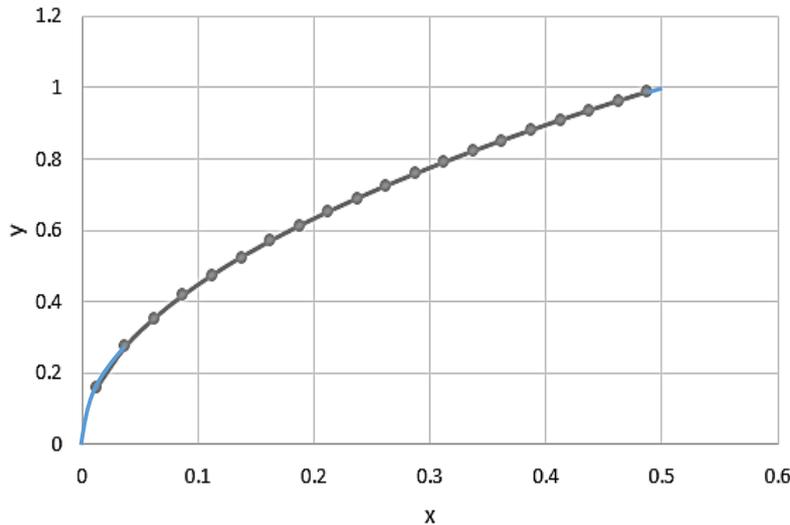
Figure 399:

profile of the central cone of the air intake fan

In discussion with the student group, it was agreed that the volume could be estimated by dividing the cone into a series of cylinders spanning the x-range from 0 to 0.5m, with the volumes then calculated and totalled using a spreadsheet:

$$volume = \sum_{0}^{0.5} \pi r^2 h$$
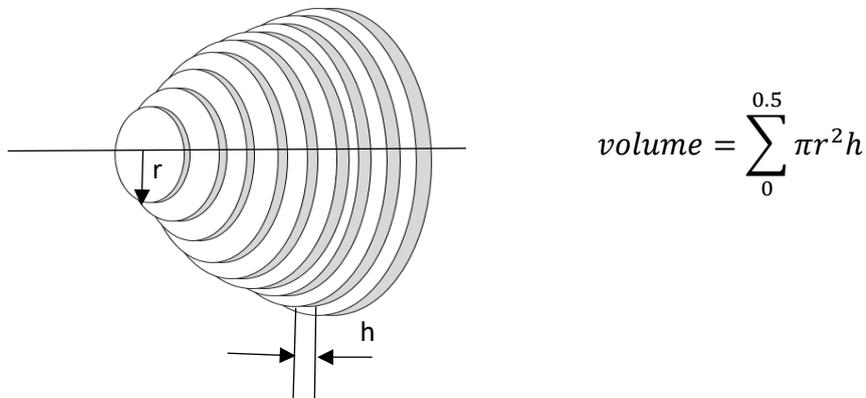
Figure 400: approximation of the cone volume as a series of discs

Results of the spreadsheet calculation are shown in figure 401 below.

The alternative analytic approach for solving this problem again carries out the integration of thin discs over the range of x-values from 0 to 0.5m:

$$\int_{0}^{0.5} \pi r^2 . dx \quad = \quad \pi \int_{0}^{0.5} \left(\sqrt{2x} \right)^2 . dx \quad = \quad \pi \int_{0}^{0.5} 2x . dx$$

π is a constant, so can be removed from the integral.

Using methods which can be found in advanced level mathematics textbooks:

$$\pi \int_{0}^{0.5} 2x \, . \, dx = \pi \, [x^2]_0^{0.5}$$

$$= \pi \, [0.5^2 - 0^2]$$

$$= 0.25\pi$$

$$= 0.7854 \, m^3$$

It is pleasing that the solution found by the numerical spreadsheet method and the analytical formula solution are in agreement to at least four decimal places. This gives us confidence that numerical methods can provide practical answers to calculus problems if an analytical approach proves to be difficult.

y	mean y	radius	volume	
0	0.0125	0.158114	0.001963	
0.025	0.0375	0.273861	0.00589	
0.05	0.0625	0.353553	0.009817	
0.075	0.0875	0.41833	0.013744	
0.1	0.1125	0.474342	0.017671	
0.125	0.1375	0.524404	0.021598	
0.15	0.1625	0.570088	0.025525	
0.175	0.1875	0.612372	0.029452	
0.2	0.2125	0.65192	0.033379	
0.225	0.2375	0.689202	0.037306	
0.25	0.2625	0.724569	0.041233	
0.275	0.2875	0.758288	0.04516	
0.3	0.3125	0.790569	0.049087	
0.325	0.3375	0.821584	0.053014	
0.35	0.3625	0.851469	0.056941	
0.375	0.3875	0.880341	0.060868	
0.4	0.4125	0.908295	0.064795	
0.425	0.4375	0.935414	0.068722	
0.45	0.4625	0.961769	0.072649	
0.475	0.4875	0.987421	0.076576	
0.5				
		total	0.785398	m³

Figure 401: numerical solution of the aircraft engine problem

Volume of a sphere

A familiar formula from GCSE mathematics is the volume of a sphere:

$$V = \frac{4}{3}\pi R^3$$

where **R** is the sphere radius. Without knowing this formula, calculus allows us to work from first principles to determine the volume of any sphere by numerical methods. Indeed, the formula itself can be derived by calculus techniques, as we will demonstrate.

To determine the volume of a sphere without the use of the volume formula, we begin by dividing the sphere into a series of thin discs in a similar way to the aircraft engine problem:
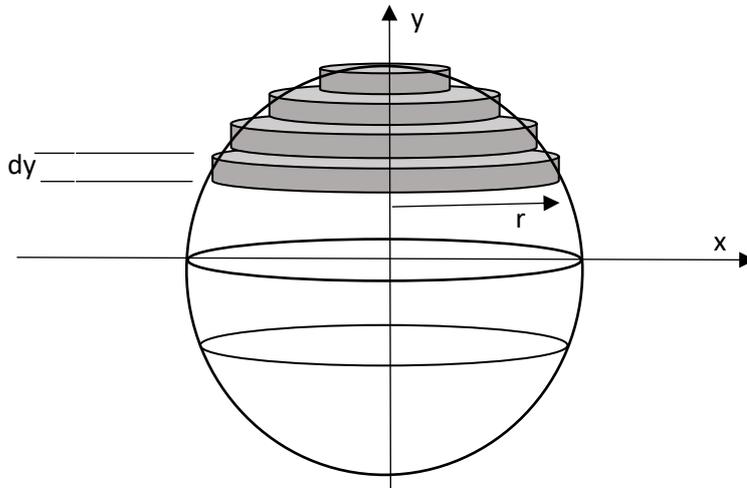
Figure 402:

representation of a sphere as a series of thin discs

As an example, we will take the radius of the sphere to be 3.5cm. A numerical solution is shown in figure 404 below. The upper hemisphere has been divided into 20 discs. The radius of each disc is taken as the mean of the radius at the upper and lower surfaces of the disc. We calculate each radius by Pythagoras' theorem:
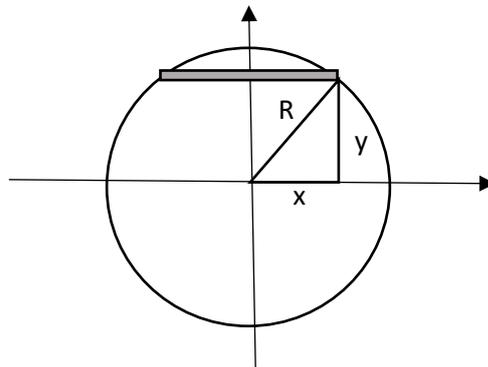
Figure 403:

determining the radius of a disc within the sphere using Pythagoras' theorem

If we specify the height **y** of the disc above the base of the hemisphere, the radius **r** of the disc is given by:

$$r = \sqrt{R^2 - y^2}$$

The volume of each disc can then be calculated as:

$$v = \pi r^2 t$$

where **t** is the disc thickness. The volumes calculated for the twenty discs are added to obtain the volume of the hemisphere, then the result is doubled to give the volume of the whole sphere.

disc	radius r	mean r	volume
0	3.500000000	3.497811	6.726368
1	3.495622262	3.489039	6.692673
2	3.482456030	3.471429	6.625282
3	3.460400988	3.444843	6.524194
4	3.429285640	3.409073	6.389407
5	3.388860428	3.363824	6.220917
6	3.338787205	3.308706	6.01872
7	3.278623949	3.243213	5.782811
8	3.207802986	3.166701	5.51318
9	3.125599942	3.078344	5.209815
10	3.031088913	2.977083	4.8727
11	2.923076290	2.861538	4.501808
12	2.800000000	2.729885	4.0971
13	2.659769727	2.579635	3.658511
14	2.499499950	2.407266	3.185929
15	2.315032397	2.207516	2.679142
16	2.100000000	1.97187	2.137688
17	1.843739407	1.684677	1.560347
18	1.525614630	1.309245	0.942388
19	1.092874650	0.546437	0.16416
20	0.000000000		
	hemisphere	volume	89.50314
	sphere	volume	179.0063

Figure 404: numerical solution for the volume of a sphere

We can compare this result of 179.01 cm^3 with the exact answer, calculated using:

$$V = \frac{4}{3}\pi R^3$$

which is 179.59 cm^3. In this instance, the numerical solution might not be sufficiently accurate. We would expect that the accuracy of the numerical solution could be improved by increasing the number of discs in the calculation, but the spreadsheet then becomes large and cumbersome. There can be advantages in obtaining exact solutions analytically in cases where a standard formula is already available, or where a formula can be easily obtained using calculus techniques. In the next section, we demonstrate how the volume formula is derived.

As in the numerical method, we consider a hemisphere to be made up of discs. The integration assumes that there are an infinite number of infinitely thin discs, so that the volume is calculated exactly. The hemisphere volume is then doubled to give the complete sphere:

$$volume\ of\ sphere = 2\int_0^R \pi r^2.dy$$

Since π is a constant, it can be moved outside of the integral:

$$V = 2\pi \int_0^R r^2.dy$$

The radius of each disc can be calculated by Pythagoras' theorem, as shown previously in figure 403:

$$r^2 = R^2 - y^2$$

Substituting for r² in the integral:

$$V = 2\pi \int_0^R R^2 - y^2 . dy$$

We now carry out the integration with respect to the variable **y**. Using calculus methods which are explained in advanced level mathematics textbooks, the integral of **R²** is **R²y**, and the integral of **y²** is **¹/₃ y³**, giving:

$$V = 2\pi \left[R^2 y - \frac{1}{3} y^3 \right]_0^R$$

A result is calculated by substituting the upper limit **R**, then the lower limit **0**, in place of the variable **y**, and subtracting the lower value from the upper value. The lower limit gives a result of zero, so:

$$V = 2\pi \left[\left(R^3 - \frac{1}{3} R^3 \right) - 0 \right]$$

$$V = 2\pi \left(R^3 - \frac{1}{3} R^3 \right)$$

$$V = 2\pi \left(\frac{2}{3} R^3 \right)$$

$$V = \frac{4}{3} \pi R^3$$

We have completed the derivation of the volume formula for a sphere.

Optimisation models

Another important application of calculus is in finding the optimum solution for problems where a range of results are possible. As a simple example, consider the following question:

> A farmer wishes to enclose a grazing area within a large field by means of a temporary electric fence. He has 200 metres of fencing material. One side of the field already has a permanent hedge, so will not need to be fenced. Determine the largest rectangular area that can be enclosed, and the lengths of its sides.
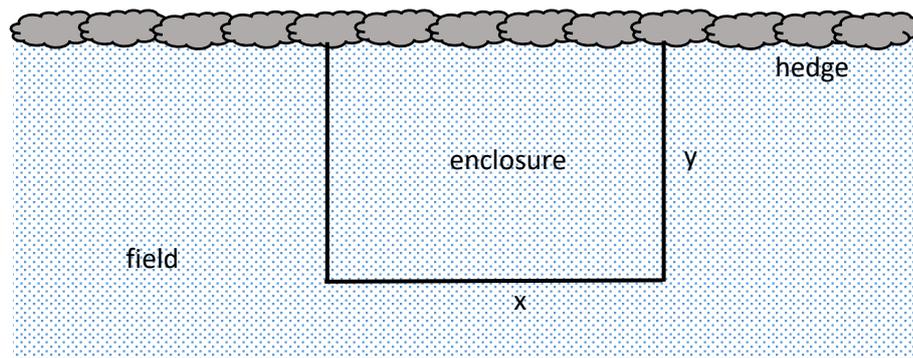
Figure 405: field enclosure optimisation problem

To solve this problem, we begin by representing the sides of the fence enclosure by the variables **x** and **y**, as shown in figure 405 above. The objective is then to maximise the area **A** of the enclosure:

$$A = x \cdot y$$

We have to take into consideration the requirement that the total length of fencing is 200 metres. The equation for the perimeter of the enclosure is therefore:

$$x + 2y = 200$$

Hence:

$$x = 200 - 2y$$

Substituting the value for x in the area formula gives:

$$A = (200 - 2y)\, y$$

$$A = 200y - 2y^2$$

This formula can be used to calculate the area of the enclosure for different values of **y**. The maximum possible length of **y** is 100 metres, leaving no fence material for the width **x** of the enclosure. Results produced by spreadsheet are shown in figure 406 below.

area = $200y - 2y^2$	
y	area
0	0
10	1800
20	3200
30	4200
40	4800
50	5000
60	4800
70	4200
80	3200
90	1800
100	0
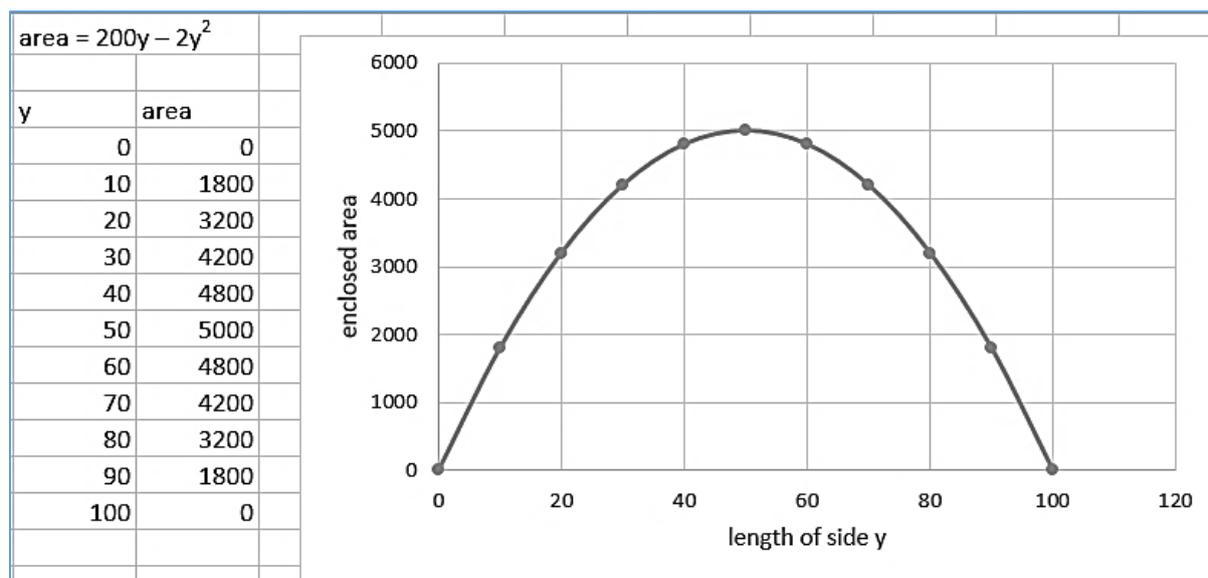
Figure 406: area of the field enclosure for different shapes of fence rectangle

The maximum area of 5 000 m² occurs when y has a length of 50m. The corresponding value for x can be calculated as:

$$200 - 2 \times 50 = 100m$$

As with most calculus problems, it is also possible to solve this question by an analytical method. Looking at the graph curve in figure 406, we notice that:

- For values of y less than the maximum, the gradient is positive, sloping uphill.
- At the maximum point, the gradient is zero with the graph line horizontal.
- For values of y greater than maximum, the gradient is negative, sloping downhill.

This forms the basis of the analytical solution method. We begin by finding the derivative of the area equation, which will tell us the gradient of the curve for any **y** value. Using the ruloes of calculus which can be found in advanced level mathematics textbooks:

$$A = 200y - 2y^2$$

$$\frac{dA}{dy} = 200 - 4y$$

The graph of the derivative is plotted by spreadsheet in figure 407 below:

y	derivative
0	200
10	160
20	120
30	80
40	40
50	0
60	-40
70	-80
80	-120
90	-160
100	-200
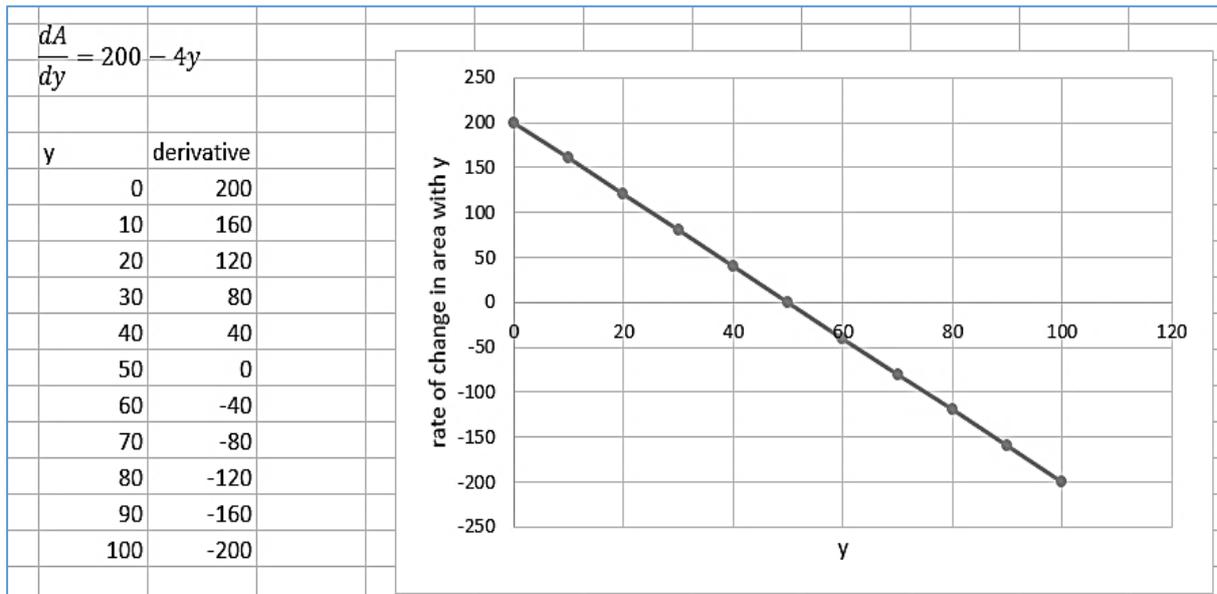
Figure 407: graph of the derivative of the area equation

We find that the graph line passes through zero at a **y** value of 50. This corresponds to the maximum point, where the gradient of the area graph was zero. The derivative curve shows positive gradient values when y is less than the maximum, and negative gradient values when y is greater than the maximum.

It is often more accurate to calculate the maximum position directly from the derivative equation using algebra, rather than by plotting a graph. We set the value of the derivative to zero at the maximum point, then solve for **y**:

$$200 - 4y = 0$$

$$4y = 200$$

$$y = 50$$

which agrees with the result we obtained previously.

The next optimisation problem involves both area and volume:

> A manufacturer needs to make a cylindrical can that will hold 1.5 litres of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.
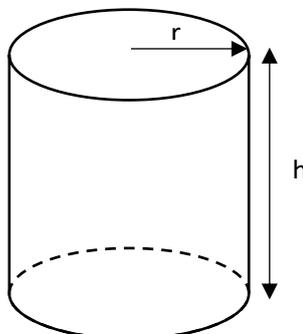
Figure 408: container to be manufactured

The can will be made up from three pieces of metal: two ends, plus a cylinder formed from a rectangle of material. We can write formulae for the areas of these components, based on the radius **r** and the height **h** of the can.
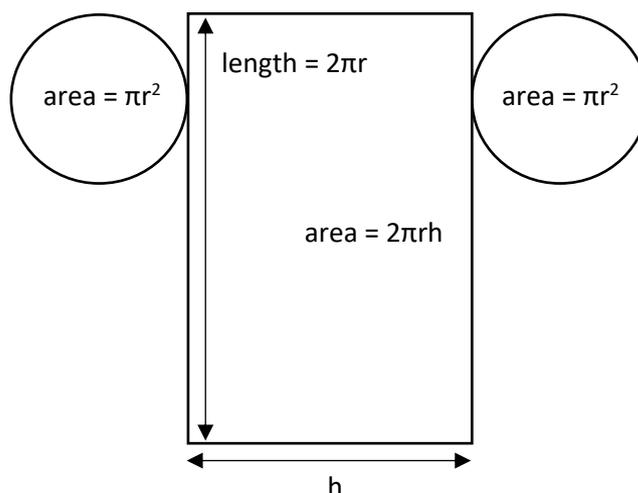
Figure 409: surface areas of components of the container

It will be convenient to work in units of centimetres. 1 litre is equivalent to 1000 cm^3. The objective is to minimise the surface area of the container:

$$A = 2\pi r^2 + 2\pi rh$$

This is subject to a requirement that the volume of the container is 1.5 litres. We can produce an equation for the volume of the cylinder, as the area of the circular base multiplied by the height **h**:

$$V = \pi r^2 h = 1500 \; cm^3$$

Rearranging this equation gives an expression for the height in terms of the cylinder radius:

$$h = \frac{1500}{\pi r^2}$$

Substituting for **h** in the area equation gives:

$$A = 2\pi r^2 + 2\pi r \left(\frac{1500}{\pi r^2}\right)$$

$$A = 2\pi r^2 + \frac{3000}{r}$$

This formula can be used to the calculate area of material needed to construct cans of different radius **r**. Results produced by spreadsheet are shown in figure 410 below.

$$A = 2\pi r^2 + \frac{3000}{r}$$

r	area
1	3006.283
2	1525.133
3	1056.549
4	850.531
5	757.0796
6	726.1947
7	736.4475
8	777.1239
9	842.2713
10	928.3185
11	1032.993
12	1154.779
13	1292.628
14	1445.79
15	1613.717
16	1795.995
17	1992.311
18	2202.419
19	2426.125
20	2663.274
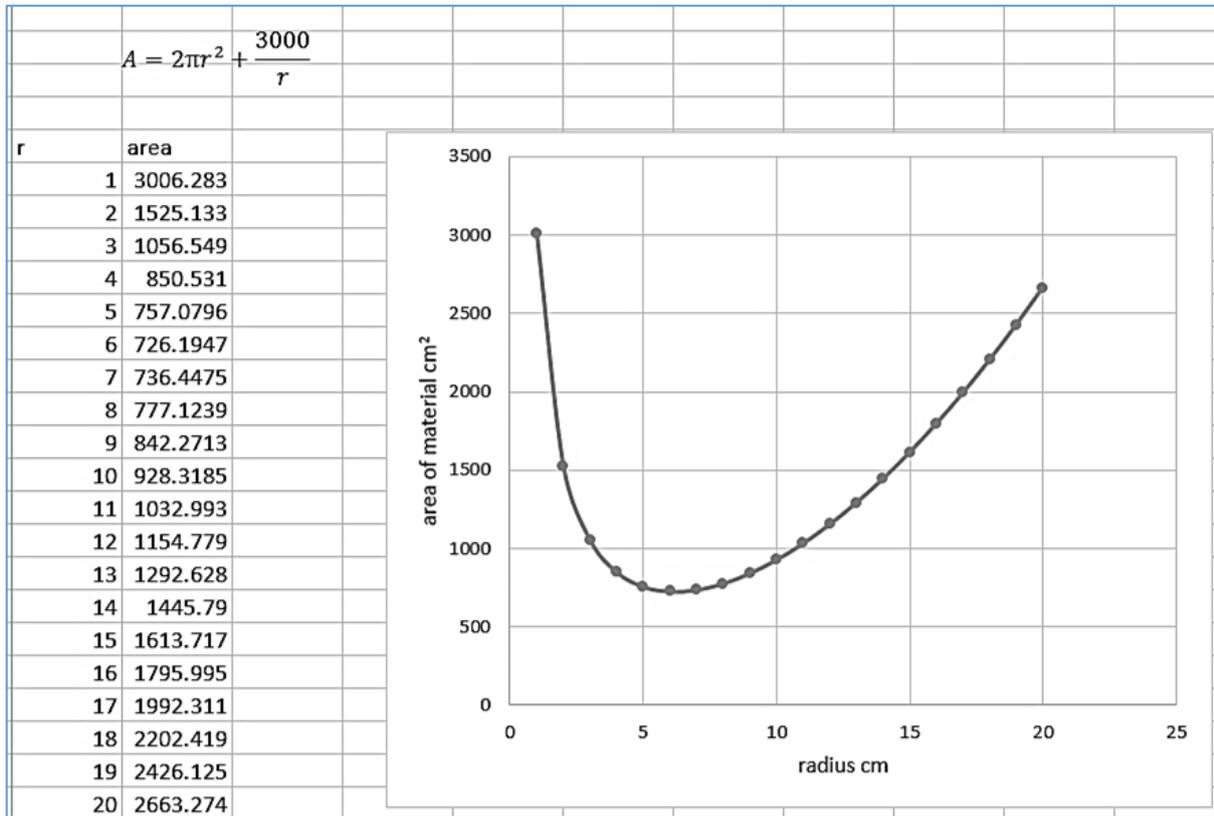
Figure 410: graph of surface area for cylindrical cans of different radius

The minimum area shown in the table of figures is 726 cm^2, corresponding to a radius of 6cm. However, we cannot be certain that the true minimum is not some other value close to this. It is difficult to identify the true minimum from the graph, as the curve is close to horizontal over the range of **r** values between 5cm and 7cm.

To obtain a more accurate solution to the problem, an analytical method can be used. Looking at the graph curve in figure 410, we notice that:

- For values of r less than the minimum, the gradient is negative, sloping downhill.
- At the minimum point, the gradient is zero with the graph line horizontal.
- For values of r greater than minimum, the gradient is positive, sloping uphill.

As in the case of the field enclosure problem earlier, we can calculate the derivative of the area formula and use this to find the point where the gradient of the area graph is zero. This will give the position of the minimum on the curve.

Using rules of calculus to find an expression for the derivative:

$$A = 2\pi r^2 + \frac{3000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{3000}{r^2}$$

The graph of the derivative is plotted by spreadsheet in figure 411 below:

$\frac{dA}{dr} = 4\pi r - \frac{3000}{r^2}$	
r	derivative
1	-2987.43
2	-724.867
3	-295.634
4	-137.235
5	-57.1681
6	-7.93511
7	26.7401
8	53.65596
9	76.0603
10	95.66371
11	113.4367
12	129.9631
13	145.6113
14	160.6231
15	175.1622
16	189.3432
17	203.2477
18	216.9354
19	230.4508
20	243.8274
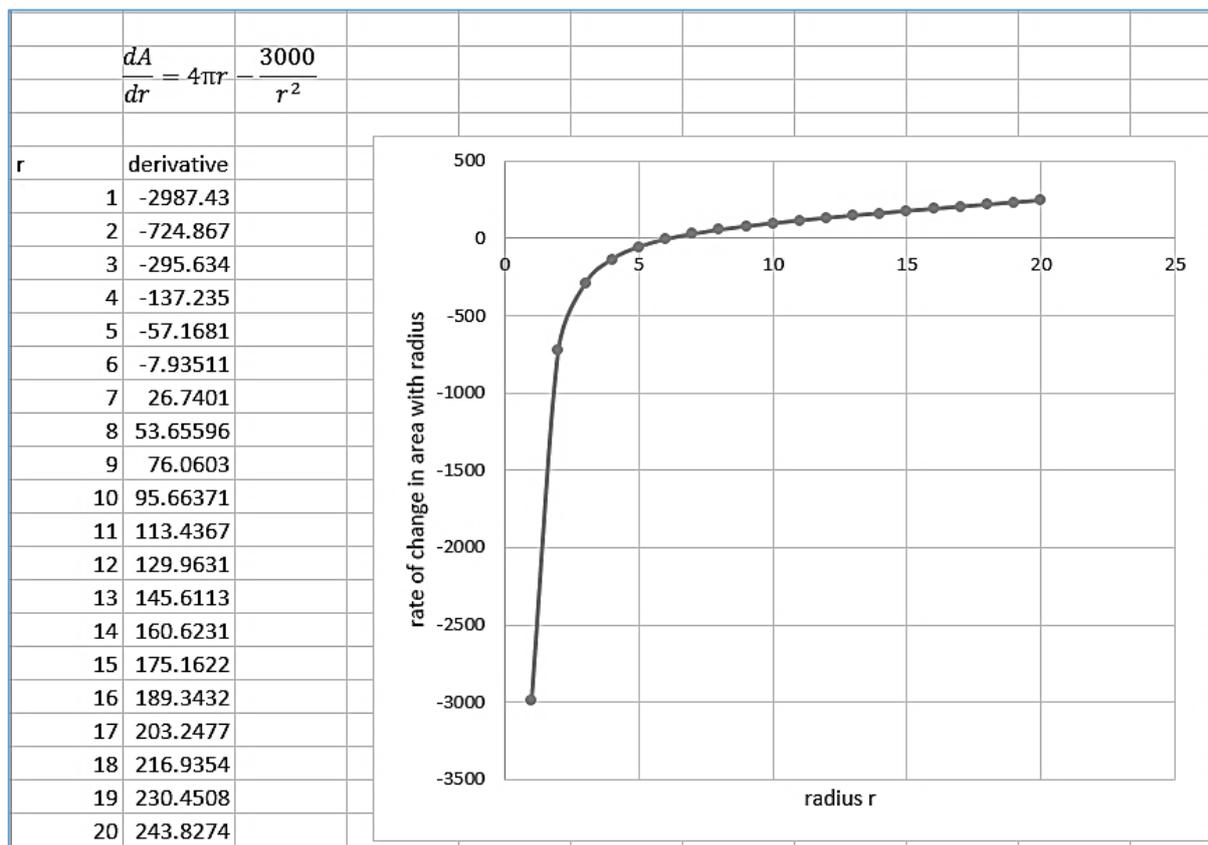
Figure 411: graph of the derivative of the area equation

We see from the table of figures that the derivative changes from negative to positive between r = 6 and r = 7, so the exact position of the minimum must lie in this interval.

At the minimum the gradient has a value of zero, so

$$4\pi r - \frac{3000}{r^2} = 0$$

Rearranging:

$$\frac{4\pi r^3 - 3000}{r^2} = 0$$

To remove the division, we can multiply both sides of the equation by r^2 :

$$4\pi r^3 - 3000 = 0$$

$$r^3 = \frac{3000}{4\pi}$$

Taking a cube root to find the result for the radius:

$$r = \sqrt[3]{\frac{3000}{4\pi}}$$

$$r = 6.20 \; cm$$

We can now find the height of this can:

$$h = \frac{1500}{\pi r^2}$$

$$h = \frac{1500}{\pi (6.20)^2}$$

$$h = 12.42 \; cm$$

The cylindrical 1.5 litre can using the minimum of material therefore has a radius of 6.2cm and a height of 12.4cm.

It is sometimes the case with optimisation problems that no algebraic formula is available to analyse, and a solution can only be found by numerical methods. As an example consider the situation shown in figure 412 below:

A group is undertaking an expedition. As part of the route, it is necessary to travel on foot from the waterfall at point A to the road junction at point B. The group must cross rough moorland to reach a forestry road, then follow this to the destination. Six possible routes A – F are marked on the map. Which would be fastest?
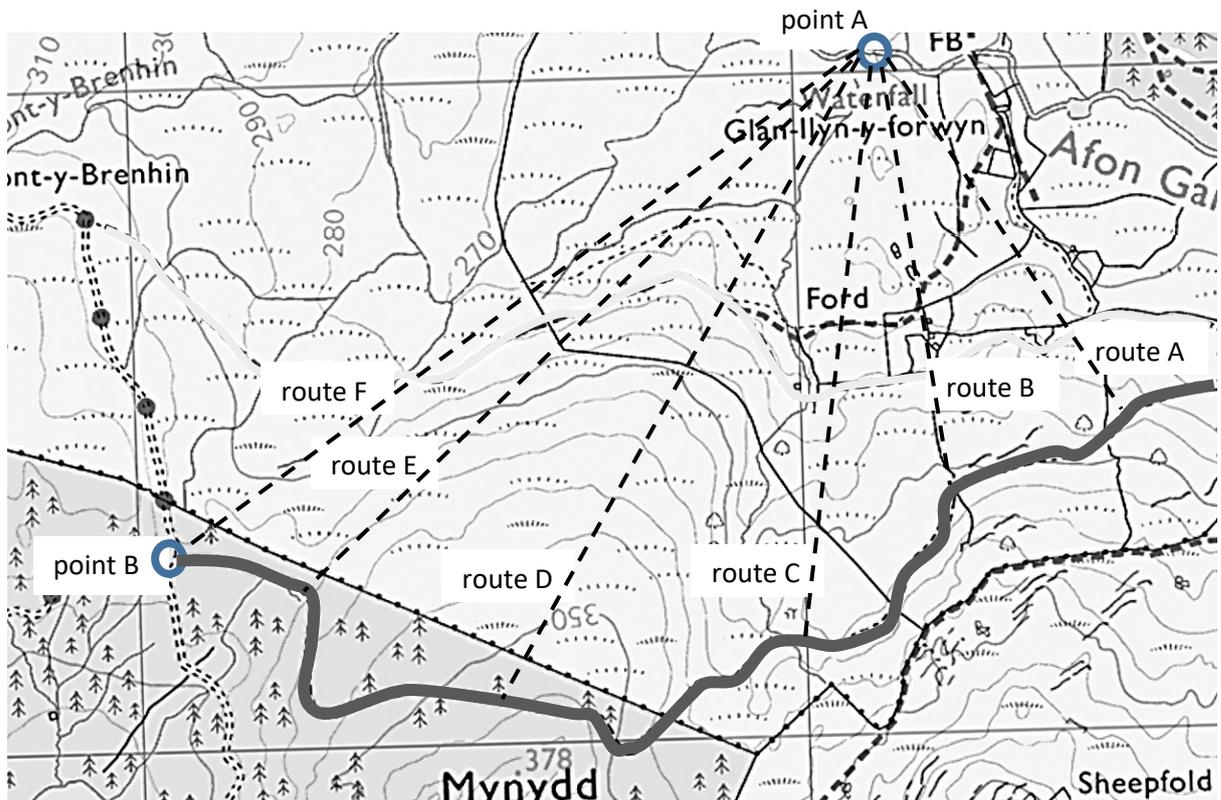
Figure 412: map of possible routes between points A and B

The group will be able to travel faster along the forestry road than across the rough moorland. From previous experience in similar terrain, it is known that the speed along the road will be approximately 4 km/h, whilst the speed across the moorland will be only 1.5 km/hour.

Measurements for the possible routes A to F are shown in figure 413.

- One possible strategy for the group is to reach the forestry road by the shortest route A. This will minimise the distance across the rough moorland at a slow pace, but will mean a longer walk along the road.
- An alternative strategy is to head directly for the destination by following route F across the moorland. This minimises the overall distance to be covered, but means a slow journey across difficult terrain.
- Routes B-E represent different degrees of compromise between the previous two strategies.
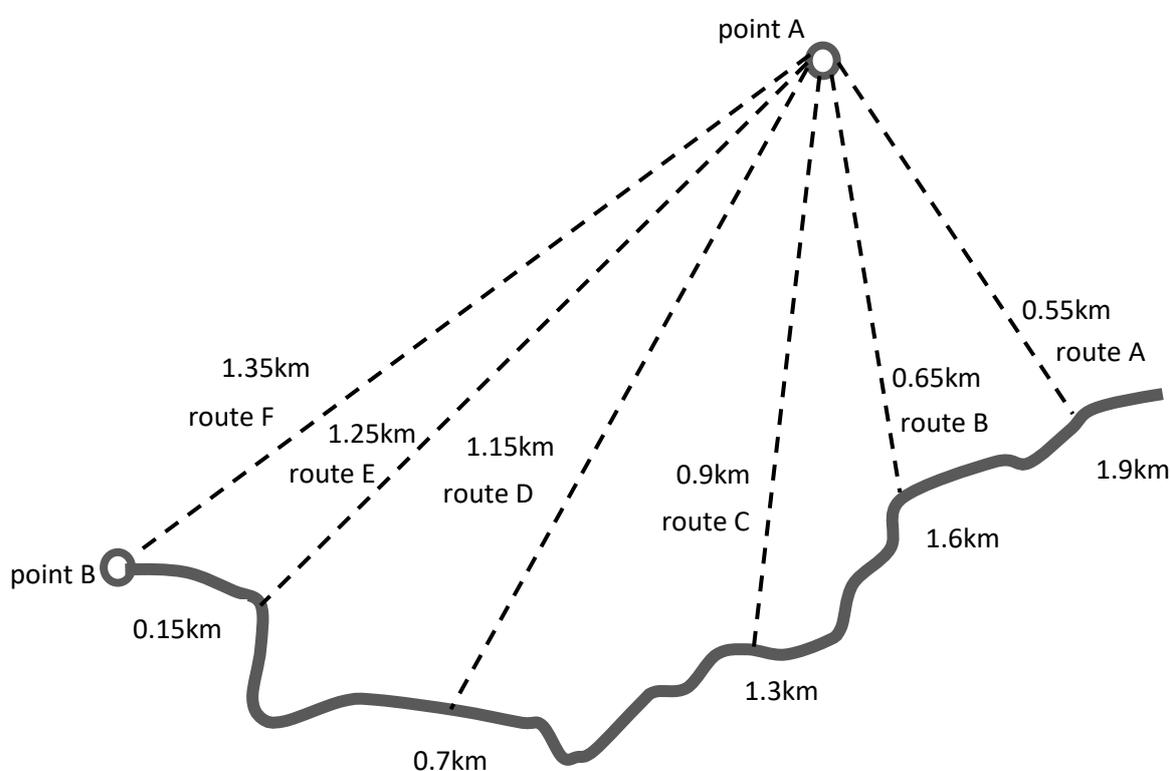
Figure 413: distances for possible routes from point A to point B

To investigate this problem, journey distances and times have been calculated by spreadsheet, and results are displayed as graphs in figure 414 below. We find that **route B** is the fastest, crossing fairly directly to the road. However, it would also be quick to take **route E** which keeps mainly to the moorland in order to minimise the overall distance travelled.

	moorland		1.5	km/h		
	road		4	km/h		
route	moorland	road	total	moorland time	road time	total time
A	0.55	1.9	2.45	0.37	0.48	0.84
B	0.65	1.6	2.25	0.43	0.40	0.83
C	0.9	1.3	2.2	0.60	0.33	0.93
D	1.15	0.7	1.85	0.77	0.18	0.94
E	1.25	0.15	1.4	0.83	0.04	0.87
F	1.35	0	1.35	0.90	0.00	0.90
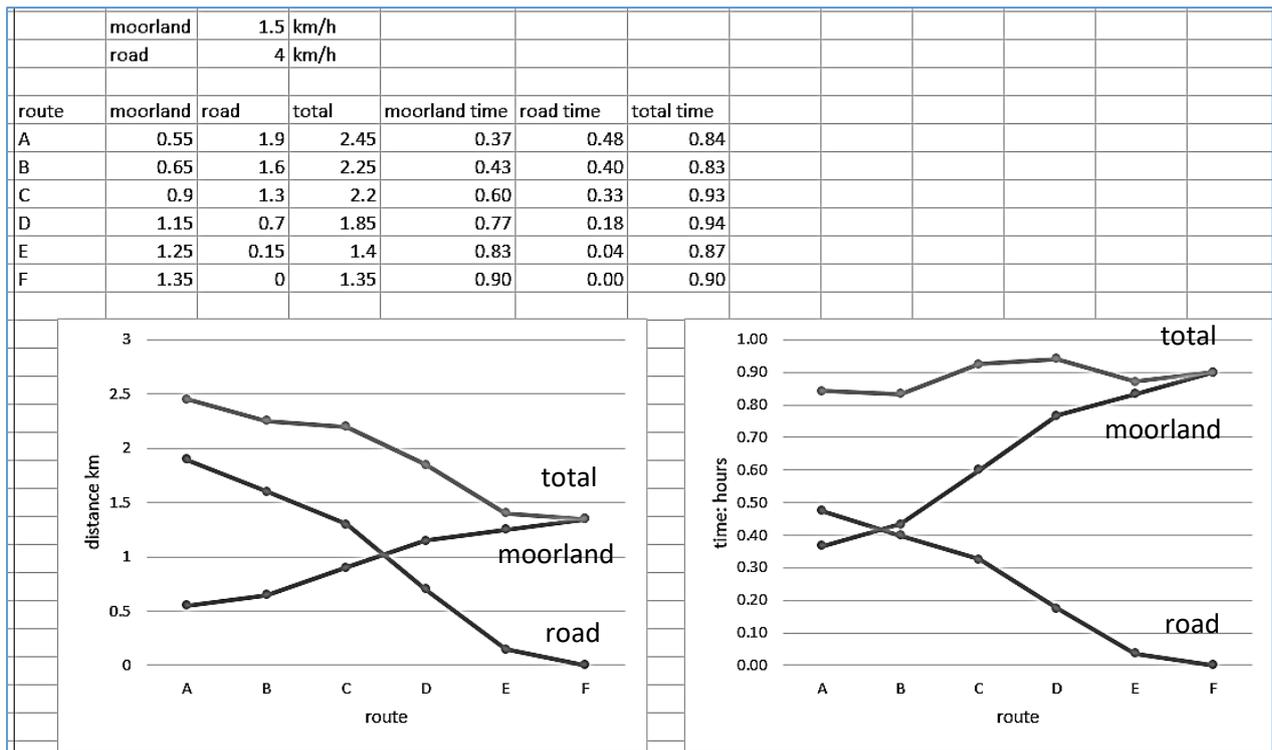
Figure 414: distances and times for the alternative routes

Summary

Calculus has two main uses:

- **Differentiation** to determine the rate at which quantities are changing. Examples are: using measurements of velocity at different intervals to determine acceleration, or using the quantity of radioactive material to determine how many atoms will decay during the next time period. Rates of change can also be useful in finding maximum and minimum values, as the rate of change becomes zero at these points.
- **Integration** to determine totals of a series of values obtained at intervals. Examples are: determining the total distance travelled from measurements of velocity at a series of points, or determining the volume of a solid shape from its cross sectional area at a series of points.

Calculus problems can often be solved using numerical methods, with results displayed as tables of figures or graphs. The alternative is to use algebraic analysis. Both of these approaches have advantages and limitations:

- Numerical solutions may be easier to understand and can be more easily checked against the real world situation. In some situations, no simple formulae may be available to describe the system, so a numerical solution is the only possibility. However, numerical solutions may only be approximate.
- Analytical solutions give precise answers, so may be necessary when high accuracy is essential. It is often quickest to use analytical solutions if a formula is available. However, finding a suitable formula may sometimes be difficult or impossible.